

A Systematic Impedance Model for Non-Linear Helmholtz Resonator Liner

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The usual non-linear corrections for a Helmholtz resonator type impedance do not seem to be based on a systematic asymptotic solution of the pertaining equations. We aim to present a systematic derivation of a solution of the non-linear Helmholtz resonator equation, in order to obtain analytically expressions for impedances close to resonance, while including non-linear effects. The amplitude regime considered is such that when we stay away from the resonance condition, the non-linear terms are relatively small and the solution obtained is of the linear equation (formed after neglecting the non-linear terms). Close to the resonance frequency, the non-linear terms can no longer be neglected and algebraic equations are obtained that describe the corresponding non-linear impedance.

I. Introduction

An important type of acoustic liner constitutes of a honeycomb array of small cells called Helmholtz resonators. The Helmholtz resonator is a cavity filled with air and having a small opening called the neck. When excited with a fluctuating external pressure, the mass of air plug inside the neck moves against the large volume of compressible air inside the cavity, which acts as a spring, while viscous forces and vortex shedding generate dissipation of energy. Altogether this establishes a mass-spring-damper system. The damping is normally relatively small such that a resonance frequency can be identified. At and near resonance, the dissipation is largest and so narrow band sound absorption is achieved for frequencies close to resonance. This process is the basic design criterion for the liners. The resonator, as “seen” from outside, is characterized by its impedance $Z = Z(\omega)$, relating (spatially averaged) pressure and velocity at the wall. Ideally, Z is a wall property and independent of the acoustic field. However, in particular near resonance Z is amplitude dependent for high but relevant amplitudes, for example of the “buzz saw” noise in a turbofan engine due to the shocks produced in front of the fan at take off and the blade tips operates in a supersonic regime [1, 2]. Since liners are designed to operate at resonance, it is important to know quantitatively and understand qualitatively such impedances Z with good precision.

The non-linear effects are mainly of hydrodynamical origin, due to vortex shedding at outflow from the opening. This is physically a rather complicated process [3–7] and the existing non-linear corrections of the impedance are either based on physically inspired modelling assumptions that do not aim to solve the equations of the non-linear resonator [8–12], or by fully CFD or DNS simulations [1, 13–18]. Only Innes and Crighton [19] analysed a non-linear Helmholtz resonator equation systematically, but their configuration is different from ours. They considered *very* high amplitudes with grazing flow and thus a (slightly) different differential equation.

The extra complication of grazing flow along the liner wall will not be considered here. It is important if the mean flow boundary layer is thin enough and the resonator outflow velocity is high but not very

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high. The present work focuses on a systematic derivation of an asymptotic solution of a stand-alone non-linear Helmholtz resonator equation from first principles.

We start with the classical modelling of the Helmholtz resonator and formulate a perturbation problem in terms of a small parameter ε which is based on the exciting amplitude. The stationary solution of this problem is solved asymptotically up to second order. Secular effects of the external forcing are treated in the usual way by a suitable Lindstedt-Poincaré type transformation. A non-standard problem was the modulus term $|u|$ of the velocity. This prohibits a standard asymptotic expansion because the location of the zero's of u are a priori unknown. This problem has been tackled by adding an unknown shift of the origin, to be determined along the construction of the solution, and using the fact that the stationary solution has the same periodicity as the driving force.

II. Mathematical Formulation

A sketch of the Helmholtz resonator considered is shown in Fig. 1. A simple and classic model (in various forms presented in the previously mentioned literature), that includes non-linear separation effects for the air flow in and out the neck, is derived as follows. If the cross-sectional area S_b of the bottle is

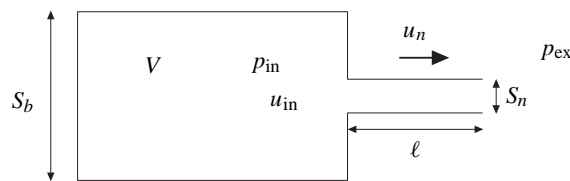


Figure 1. Helmholtz resonator

large compared to the cross sectional area S_n of the neck, the acoustic velocities in the bottle will be small compared to those in the neck. Hence we may assume that the pressure and density perturbations p_{in} and ρ_{in} in the bottle are uniform. If the cavity neck is acoustically compact, *i.e.* $kl \ll 1$ for a typical wave number $k = \omega/c_0$, we can neglect compressibility in the neck and integrate along a cross section S_n the line integral of the momentum equation

$$\rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla p = \mu \nabla^2 \mathbf{v}$$

along a streamline from a point inside to a point outside to obtain for the cross sectionally averaged velocity $\bar{\mathbf{v}}$ the relation

$$\rho_0 \int_{in}^{ex} \frac{\partial \bar{\mathbf{v}}}{\partial t} \cdot \mathbf{ds} + \frac{1}{2} \rho_0 (u_{ex}^2 - u_{in}^2) + (p_{ex} - p_{in}) = \int_{in}^{ex} \mu \overline{\nabla^2 \mathbf{v}} \cdot \mathbf{ds}. \quad (1)$$

Assuming that the streamline does not change in time (for example the center streamline) we have

$$\int_{in}^{ex} \frac{\partial \bar{\mathbf{v}}}{\partial t} \cdot \mathbf{ds} = \frac{d}{dt} \int_{in}^{ex} \bar{\mathbf{v}} \cdot \mathbf{ds}. \quad (2)$$

The velocity line integral evidently scales on a typical length times a typical velocity. If end effects are minor, we can use the neck flux velocity u_n with a corresponding length being the neck length ℓ , added by a small end correction δ to take into account the inertia of the acoustic flow at both ends just outside the neck (inside and outside the resonator). Then we have

$$\int_{in}^{ex} \bar{\mathbf{v}} \cdot \mathbf{ds} = (\ell + 2\delta) u_n. \quad (3)$$

For the stress term line integral we observe that, apart from u_n itself, it will depend on flow profile, Reynolds number, wall heat exchange, turbulence, separation from sharp edges, and maybe more. Following Melling [9], we will take these effects together in a resistance factor R , which will a priori be assumed to be relatively small, in order to have resonance and a small decay per period.

$$\int_{\text{in}}^{\text{ex}} \mu \overline{\nabla^2 \mathbf{v}} \cdot \mathbf{ds} = -R u_n. \quad (4)$$

(Note that this form is exact for a Poiseuille flow with parabolic profile). Due to separation from the outer exit, we have with outflow $u_{\text{in}} \simeq 0$ with $u_{\text{ex}} = u_n$ jetting out, while similarly during inflow, $u_{\text{ex}} \simeq 0$ with $u_{\text{in}} = u_n$ jetting into the cavity. The pressure in the jets, however, has to remain equal to the surrounding pressure (p_{ex} and p_{in} respectively) because the boundary of the jet cannot support a pressure difference. Therefore, we have altogether

$$\rho_0(\ell + 2\delta) \frac{d}{dt} u_n + \frac{1}{2} \rho_0 u_n |u_n| + R u_n = p_{\text{in}} - p_{\text{ex}}. \quad (5)$$

The second equation between p_n and u_n is obtained by applying the integral mass conservation law on the volume V of the cavity. The change of mass must be equal to the flux through the cavity neck, which is in linearised form for the density perturbation ρ_{in}

$$V \frac{d\rho_{\text{in}}}{dt} = -\rho u_n S_n \approx -\rho_0 u_n S_n. \quad (6)$$

Assuming an adiabatic compression of the fluid in the cavity, we have $p_{\text{in}} = c_0^2 \rho_{\text{in}}$. Elimination of ρ_{in} and u_n from (5) by using (6) and redefining $(\ell + 2\delta) := \ell$ yields the non-linear Helmholtz resonator equation

$$\frac{\ell V}{c_0^2 S_n} \frac{d^2 p_{\text{in}}}{dt^2} + \frac{V^2}{2\rho_0 c_0^4 S_n^2} \frac{d p_{\text{in}}}{dt} \left| \frac{d p_{\text{in}}}{dt} \right| + \frac{R V}{\rho_0 c_0^2 S_n} \frac{d p_{\text{in}}}{dt} + p_{\text{in}} = p_{\text{ex}}. \quad (7)$$

For a proper analysis it is most clarifying to rewrite the equation into non-dimensional variables. For this we need an inherent timescale and pressure level. For vanishing amplitudes and negligible dissipation the equation describes a harmonic oscillator, so the reciprocal of its angular frequency

$$\omega_0 = \frac{c_0}{\ell} \left(\frac{\ell S_n}{V} \right)^{1/2}$$

is a suitable timescale of the problem. By dividing the non-linear damping term by the acceleration term we find the pressure level $2\rho_0 c_0^2 \ell S_n / V$ at which the non-linear damping would be just as large as the other terms. So for a pressure that is a small fraction ε of this level we have a problem with only little non-linear damping. In addition we assume that the linear damping is small and of the same order of magnitude as the non-linear damping (that is to say: near resonance. Away from resonance the nonlinear term will be relatively smaller). Also, the driving amplitude p_{ex} will be an order smaller than p_{in} . In order to make all this explicit we introduce a small parameter (via the external forcing amplitude), and make dimensionless

$$\tau = \omega_0 t, \quad p_{\text{in}} = 2\varepsilon \rho_0 c_0^2 \left(\frac{\ell S_n}{V} \right) y, \quad p_{\text{ex}} = 2\varepsilon^2 \rho_0 c_0^2 \left(\frac{\ell S_n}{V} \right) F, \quad R = \varepsilon \rho_0 c_0 \left(\frac{\ell S_n}{V} \right)^{1/2} r, \quad (8)$$

where $0 < \varepsilon \ll 1$ and $r, y, F = O(1)$.

Suppose that we excite the Helmholtz resonator harmonically, such that $p_{\text{ex}} = C \cos(\omega t) + H(t)$ consists of a time-harmonic component of frequency ω plus a small contribution of higher harmonics H due to the interaction with the resonator¹. In the scaled variables τ and F this becomes

$$F = F_0 \cos(\Omega \tau) + \varepsilon^{\nu} h(\tau), \quad \Omega = \frac{\omega}{\omega_0}, \quad (9)$$

¹ H will play no role in the results, but appears from the linear application of section V. Here, the external forcing field is a combination of incident and reflected waves, say $p_{\text{ex}} = f(t) + g(t)$ and $v_{\text{ex}} \propto f(t) - g(t)$. If incident part f is harmonic, reflected part g will be harmonic plus higher harmonics. We will see, however, that these higher harmonics are one or two orders of magnitude smaller, and therefore play no role in y_1 , eq. (31), resp. Y_1 , eq. (16).

where $\nu = 1$ in the resonant case and $\nu = 2$ in the non-resonant case.

Note that ε is a bookkeeping parameter, meant to measure the “smallness” of the various parameters and variables. In practice it is determined by the external forcing p_{ex} , so in the simple case of a harmonic excitation we can take $F_0 = 1$, and this will be done in any example below.

Finally we arrive at the weakly nonlinear forced oscillator as given by (10). The initial conditions are not important as we are interested only in the stationary state² of the oscillator synchronised with the forcing.

$$\frac{d^2 y}{d\tau^2} + \varepsilon \frac{dy}{d\tau} \left| \frac{dy}{d\tau} \right| + \varepsilon r \frac{dy}{d\tau} + y = \varepsilon F_0 \cos(\Omega\tau) + \varepsilon^{1+\nu} h(\tau). \quad (10)$$

We note in passing that the problem considered by Innes and Crighton [19] relates to ours if we replace $y'|y'|$ by $y'|y|$, assume $y = O(\varepsilon^{-2})$ and $F = O(\varepsilon^{-4})$, and neglect r .

III. Asymptotic Analysis

III.A. Non-Resonant Case

Away from resonance, when $1 - \Omega^2 = O(1)$, the perturbation problem is regular and relatively straightforward. We will include it here for reference.

We look for solutions of

$$y'' + \varepsilon y'|y'| + \varepsilon r y' + y = \varepsilon F_0 \cos(\Omega\tau) + \varepsilon^3 h(\tau) \quad (11)$$

that are only caused by the external forcing. Since this forcing term is $O(\varepsilon)$ and we are not near resonance, the response is of the same order of magnitude, and we transform $y = \varepsilon Y$, where $Y = O(1)$.

$$Y'' + \varepsilon^2 Y'|Y'| + \varepsilon r Y' + Y = F_0 \cos(\Omega\tau) + \varepsilon^2 h(\tau). \quad (12)$$

After substituting the assumed expansion $Y(\tau; \varepsilon) = Y_0(\tau) + \varepsilon Y_1(\tau) + \varepsilon^2 Y_2(\tau) + \dots$ and collecting the coefficients of $O(1)$, we have

$$Y_0'' + Y_0 = F_0 \cos(\Omega\tau). \quad (13)$$

The solution that follows the driving force is periodic with frequency Ω and so

$$Y_0 = \frac{F_0}{1 - \Omega^2} \cos(\Omega\tau), \quad (14)$$

Next we collect the coefficients of $O(\varepsilon)$ to obtain

$$Y_1'' + Y_1 = -r Y_0' = \frac{r F_0 \Omega}{1 - \Omega^2} \sin(\Omega\tau) \quad (15)$$

with solution

$$Y_1 = \frac{r F_0 \Omega}{(1 - \Omega^2)^2} \sin(\Omega\tau). \quad (16)$$

We may go on to $O(\varepsilon^2)$ and find the appearance of higher harmonics. Efficiently collecting terms together, we obtain for the full solution

$$y = \varepsilon F_0 \frac{(1 - \Omega^2) \cos \Omega t + \varepsilon r \Omega \sin \Omega t}{(1 - \Omega^2)^2 + \varepsilon^2 r^2 \Omega^2} + O(\varepsilon^3) \quad (17)$$

showing that the response is indeed $O(\varepsilon)$ and follows the excitation almost in phase ($1 - \Omega^2 > 0$) or anti-phase ($1 - \Omega^2 < 0$). This is not the case anymore near resonance when $1 - \Omega^2 = O(\varepsilon)$.

²In the appendix it is proved that solutions of this equation (10) are stable, so the stationary solution exists.

III.B. Resonant Case

Near resonance when $1 - \Omega^2 = O(\varepsilon)$, it was assumed and indeed confirmed by (17) that the amplitude y rises to levels of $O(1)$, and the assumption that the non-linear damping is negligible to leading orders is not correct. As the physics of the problem essentially change when $\Omega = 1 + O(\varepsilon)$, we introduce a parameter $\Delta = O(1)$ and assume that

$$\Omega = 1 + \varepsilon\Delta. \quad (18)$$

However, posed in this form we obtain secular terms in the expansion $\cos(\tau + \varepsilon\Delta\tau) = \cos(\tau) - \varepsilon\Delta\tau \sin(\tau) + \dots$ of the driving force, which prohibits a uniform approximation of y later [20, sec 15.3.2]. Therefore we remove the ε -dependence from the driving force by absorbing Ω into a new time coordinate.

Moreover, the asymptotic expansion of the modulus $|y'|$ introduces difficulties near the ε -dependent (and unknown) zero's of y' . This will be tackled by a translation of the origin by an amount $\theta(\varepsilon)$, such that the locations of the sign change of y' are fixed (as y is synchronised with the driving force) and independent of ε . (Of course, a certain amount of smoothness is anticipated such that y' has the same number of zero's per period as the forcing term). So we introduce

$$\tilde{\tau} = \Omega\tau - \theta(\varepsilon)$$

to obtain

$$\Omega^2 \frac{d^2 y}{d\tilde{\tau}^2} + \varepsilon\Omega^2 \frac{dy}{d\tilde{\tau}} \left| \frac{dy}{d\tilde{\tau}} \right| + \varepsilon\Omega r \frac{dy}{d\tilde{\tau}} + y = \varepsilon F_0 \cos(\tilde{\tau} + \theta) + \varepsilon^2 h, \quad (19)$$

where θ is to be chosen such that $y'(\tilde{\tau}) = 0$ at $\tilde{\tau} = N\pi$. In other words, $\Omega\tau = \omega t = \theta$ corresponds with the phase lag of response p_{in} to excitation p_{ex} .

When we substitute the following (assumed uniform) asymptotic expansions for y and θ [21]

$$y(\tilde{\tau}; \varepsilon) = y_0(\tilde{\tau}) + \varepsilon y_1(\tilde{\tau}) + \varepsilon^2 y_2(\tilde{\tau}) + \dots, \quad \text{and} \quad \theta(\varepsilon) = \theta_0 + \varepsilon\theta_1 + \dots,$$

and collect like powers of ε , we find for y_0

$$\frac{d^2 y_0}{d\tilde{\tau}^2} + y_0 = 0, \quad y_0'(N\pi) = 0. \quad (20)$$

This has the general solution

$$y_0(\tilde{\tau}) = A_0 \cos(\tilde{\tau}), \quad (21)$$

with A_0 and θ_0 to be determined. Although y_0 is the result of driving force F , at this level we don't have any information about their relation yet, so we can't determine the integration constants A_0 and θ_0 . Therefore, we continue with the next order y_1 .

$$\begin{aligned} \frac{d^2 y_1}{d\tilde{\tau}^2} + y_1 &= F_0 \cos(\tilde{\tau} + \theta_0) - 2\Delta \frac{d^2 y_0}{d\tilde{\tau}^2} - \frac{dy_0}{d\tilde{\tau}} \left| \frac{dy_0}{d\tilde{\tau}} \right| - r \frac{dy_0}{d\tilde{\tau}} \\ &= F_0 \cos(\tilde{\tau} + \theta_0) + 2\Delta A_0 \cos(\tilde{\tau}) + A_0 |A_0| \sin(\tilde{\tau}) |\sin(\tilde{\tau})| + r A_0 \sin(\tilde{\tau}) \end{aligned} \quad (22)$$

From the arguments that y is the stationary solution and its asymptotic expansion is uniform in $\tilde{\tau}$, it follows that no resonant excitation is allowed in the right hand side of the equation for y_1 . This means that we should suppress the cos- and sin-terms, including the first term of the Fourier expansion of

$$\sin(\tilde{\tau}) |\sin(\tilde{\tau})| = -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\tilde{\tau}}{(n^2 - \frac{1}{4})(n + \frac{3}{2})} = \frac{8}{3\pi} \sin \tilde{\tau} + \dots \quad (23)$$

to obtain

$$F_0 \cos \theta_0 = -2\Delta A_0, \quad F_0 \sin \theta_0 = \left(\frac{8}{3\pi} |A_0| + r \right) A_0 \quad (24)$$

or

$$\left[\left(\frac{8}{3\pi} |A_0| + r \right)^2 + (2\Delta)^2 \right] A_0^2 = F_0^2, \quad \tan(\theta_0) = -\frac{\frac{8}{3\pi} |A_0| + r}{2\Delta}. \quad (25)$$

In general, the equation for A_0 has to be solved numerically, from which θ_0 follows. There exist two (real) solutions, while if (A_0, θ_0) is a solution, then also $(-A_0, \theta_0 + \pi)$. So, if convenient, we could assume that A_0 is positive and maintain $|A_0| = A_0$, but this depends on θ_0 .

The next order y_1 is then given by

$$y_1(\tilde{\tau}) = A_1 \cos \tilde{\tau} + B_1 \sin \tilde{\tau} + \frac{1}{4\pi} A_0 |A_0| \sum_{n=1}^{\infty} \frac{\sin(2n+1)\tilde{\tau}}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})} \quad (26)$$

with derivative

$$y_1'(\tilde{\tau}) = -A_1 \sin \tilde{\tau} + B_1 \cos \tilde{\tau} + \frac{1}{4\pi} A_0 |A_0| \sum_{n=1}^{\infty} \frac{(2n+1) \cos(2n+1)\tilde{\tau}}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})} \quad (27)$$

and so the boundary condition

$$\begin{aligned} y_1'(N\pi) &= -A_1 \sin(N\pi) + B_1 \cos(N\pi) + \frac{1}{4\pi} A_0 |A_0| \sum_{n=1}^{\infty} \frac{(2n+1) \cos((2n+1)N\pi)}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})} \\ &= (-1)^N B_1 + \frac{(-1)^N}{4\pi} A_0 |A_0| \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})} = (-1)^N \left[B_1 + \frac{2}{9\pi} A_0 |A_0| \right] = 0 \end{aligned} \quad (28)$$

is satisfied by

$$B_1 = -\frac{2}{9\pi} A_0 |A_0|, \quad \text{because} \quad \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})} = \frac{8}{9}.$$

The sum of the telescoping series is easily found by partial fractions and noting the terms cancelling in pairs. Altogether we have

$$y_1(\tilde{\tau}) = A_1 \cos \tilde{\tau} - \frac{2}{9\pi} A_0 |A_0| \sin \tilde{\tau} + \frac{1}{4\pi} A_0 |A_0| \sum_{n=1}^{\infty} \frac{\sin(2n+1)\tilde{\tau}}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})}. \quad (29)$$

The amplitude A_1 is to be determined in a similar way as with y_0 by suppressing resonant terms in y_2 . The next order term y_2 is obtained from (19) when it is expanded to $O(\varepsilon^2)$ and terms of $O(\varepsilon^2)$ are collected

$$y_2'' + y_2 = -\Delta^2 y_0'' - 2\Delta y_1'' - 2\Delta y_0' |y_0'| - 2y_1' |y_0'| - r y_1' - r \Delta y_0' - \theta_1 F_0 \sin(\tilde{\tau} + \theta_0) + h. \quad (30)$$

After substituting y_0 and y_1 , and considering only the terms on the right hand side that are possibly in resonance with the left hand side, we obtain

$$\begin{aligned} y_2'' + y_2 &= \Delta^2 A_0 \cos \tilde{\tau} + 2\Delta A_1 \cos \tilde{\tau} - \frac{4}{9\pi} \Delta A_0 |A_0| \sin \tilde{\tau} \\ &+ \left(2\Delta A_0 \sin \tilde{\tau} + 2A_1 \sin \tilde{\tau} + \frac{4}{9\pi} A_0 |A_0| \cos \tilde{\tau} - \frac{1}{\pi} A_0 |A_0| \sum_{n=1}^{\infty} \frac{\cos(2n+1)\tilde{\tau}}{n(n+1)(n - \frac{1}{2})(n + \frac{3}{2})} \right) |A_0 \sin \tilde{\tau}| \\ &+ r A_1 \sin \tilde{\tau} + \frac{2}{9\pi} r A_0 |A_0| \cos \tilde{\tau} + r \Delta A_0 \sin \tilde{\tau} - \theta_1 F_0 \cos \theta_0 \sin \tilde{\tau} - \theta_1 F_0 \sin \theta_0 \cos \tilde{\tau} + \dots \end{aligned} \quad (31)$$

By Fourier expansion it can be found that

$$\cos \tilde{\tau} | \sin \tilde{\tau} | = \frac{4}{3\pi} \cos \tilde{\tau} + \dots, \quad \sum_{n=1}^{\infty} \frac{|\sin \tilde{\tau}| \cos(2n+1)\tilde{\tau}}{n(n+1)(n - \frac{1}{2})(n + \frac{3}{2})} = \frac{1}{\pi} \left(\frac{80}{27} - \frac{\pi^2}{3} \right) \cos \tilde{\tau} + \dots \quad (32)$$

and only higher harmonics otherwise. Suppressing the cos- and sin-terms of (31) thus results into

$$\begin{aligned} 2\Delta A_1 - \theta_1 F_0 \sin \theta_0 &= -\Delta^2 A_0 - \left(\frac{1}{3} - \frac{64}{27\pi^2}\right) A_0^3 - \frac{2}{9\pi} r A_0 |A_0| \\ \left(\frac{16}{3\pi} |A_0| + r\right) A_1 - \theta_1 F_0 \cos \theta_0 &= -\left(\frac{44}{9\pi} |A_0| + r\right) \Delta A_0 \end{aligned} \quad (33)$$

By solving the linear system (33), we can obtain A_1 and θ_1 .

IV. Time-domain Solution

The solution $y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$ ascertains in principle (for small ε) a better approximation of y than the leading order approximation y_0 , which would later provide a better approximation of the impedance. We have this full solution as

$$y(\tilde{\tau}; \varepsilon) = (A_0 + \varepsilon A_1) \cos \tilde{\tau} - \frac{2}{9\pi} \varepsilon A_0 |A_0| \sin \tilde{\tau} + \frac{1}{4\pi} \varepsilon A_0 |A_0| \sum_{n=1}^{\infty} \frac{\sin(2n+1)\tilde{\tau}}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})} + \dots \quad (34)$$

where $\tilde{\tau} = \omega t - \theta$ and $\theta = \theta_0 + \varepsilon \theta_1 + \dots$. The constants A_0, θ_0 and A_1, θ_1 can be determined from (24) and (33) respectively.

Consider first the leading order approximation. Equation (25) for A_0 has 2 real symmetric solutions (of which we normally need to consider only the positive one), but solving $A_0 = A_0(\Delta)$ is not straightforward. Therefore, it is useful to consider the inverse, $\Delta = \Delta(A_0)$, given by

$$4\Delta^2 = \frac{F_0^2}{A_0^2} - \left(\frac{8}{3\pi} |A_0| + r\right)^2 \quad (35)$$

Since $\Delta^2 \geq 0$ we see immediately that solutions exist only for a finite interval in A_0 , while $\Delta \rightarrow \infty$ only when $A_0 \rightarrow 0$. In particular, we have

$$A_0 \simeq \frac{F_0}{2|\Delta|}, \quad \tan \theta_0 \simeq -\frac{r}{2\Delta} \quad \text{or} \quad \theta_0 \simeq -\frac{r}{2\Delta} + n\pi, \quad (36)$$

which is in exact agreement with the asymptotic behaviour for $\Omega = 1 + \varepsilon \Delta$, Δ large, corresponding to the linear solution (14). In fact, by tracing the solution parametrically as a function of Δ , we can see that if we start with $\theta_0 = 0$ for $\Delta \rightarrow -\infty$, we end with $\theta_0 = \pi$ for $\Delta \rightarrow \infty$. This way, we have obtained the expression for A_0 and θ_0 ; see Fig. 2 for an example.

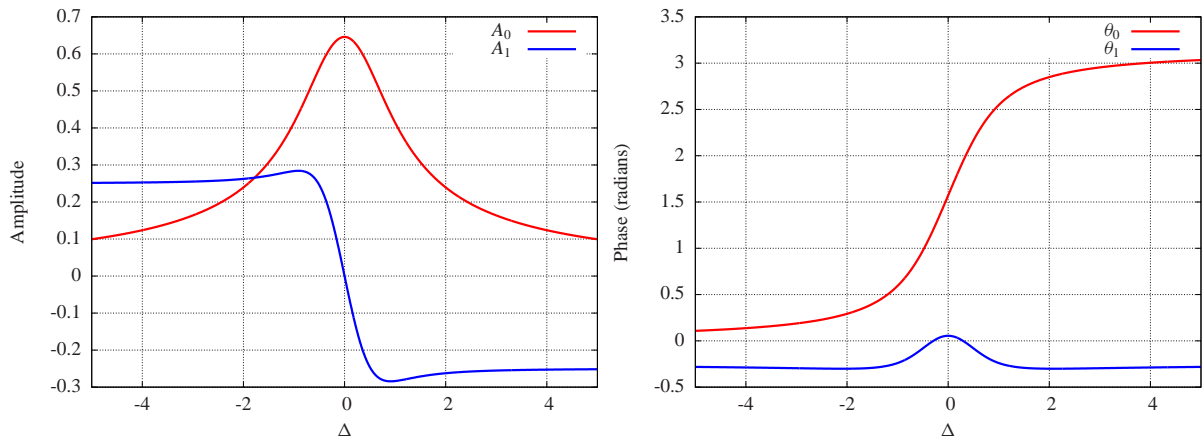


Figure 2. Solution of amplitude (A_0, A_1) and phase (θ_0, θ_1) as a function of Δ , while $r = F_0 = 1$

Substituting the obtained value of A_0 and θ_0 in (33), we can solve the linear algebraic system to obtain A_1 and θ_1 . This way, we have determined all the coefficients in (34); hence, the solution y is known which, when used with (8), gives p_{in}

$$p_{\text{in}} = 2\varepsilon\rho_0c_0^2\frac{\ell S_n}{V}\left[(A_0 + \varepsilon A_1)\cos(\omega t - \theta) - \frac{2}{9\pi}\varepsilon A_0|A_0|\sin(\omega t - \theta) + \frac{1}{4\pi}\varepsilon A_0|A_0|\sum_{n=1}^{\infty}\frac{\sin(2n+1)(\omega t - \theta)}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})} + \dots\right]. \quad (37)$$

From this solution and (6) we may determine the neck velocity u_n

$$u_n = 2\varepsilon\omega\ell\left[(A_0 + \varepsilon A_1)\sin(\omega t - \theta) + \frac{2}{9\pi}\varepsilon A_0|A_0|\cos(\omega t - \theta) - \frac{1}{4\pi}\varepsilon A_0|A_0|\sum_{n=1}^{\infty}\frac{(2n+1)\cos(2n+1)(\omega t - \theta)}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})} + \dots\right], \quad (38)$$

which will be used to obtain the impedance of the resonator in a later section.

IV.A. Comparison in Time-domain with a Fully Numerical Solution

The solution (34), correct till $O(\varepsilon)$ (y_0) and $O(\varepsilon^2)$ ($y_0 + \varepsilon y_1$), are compared with a fully numerical solution of (10), obtained by Mathematica with a standard Runge-Kutta routine, see Fig. 3. In both cases $r = 0.2$, $\Delta = 1$, while $\varepsilon = 0.28$ in the left figure and $\varepsilon = 0.88$ in the right. Note that this last case is added to see how the solution behaves for values of ε that are really not small anymore. The one with $\varepsilon = 0.28$ is indeed remarkably accurate for $y_0 + \varepsilon y_1$, and we may observe an error of y_0 and $y_0 + \varepsilon y_1$ compared to y that follows indeed the predicted behaviour of $O(\varepsilon)$ and $O(\varepsilon^2)$. The one with $\varepsilon = 0.88$ cannot be expected to be really accurate, but surprisingly the results are still of the right order of magnitude.

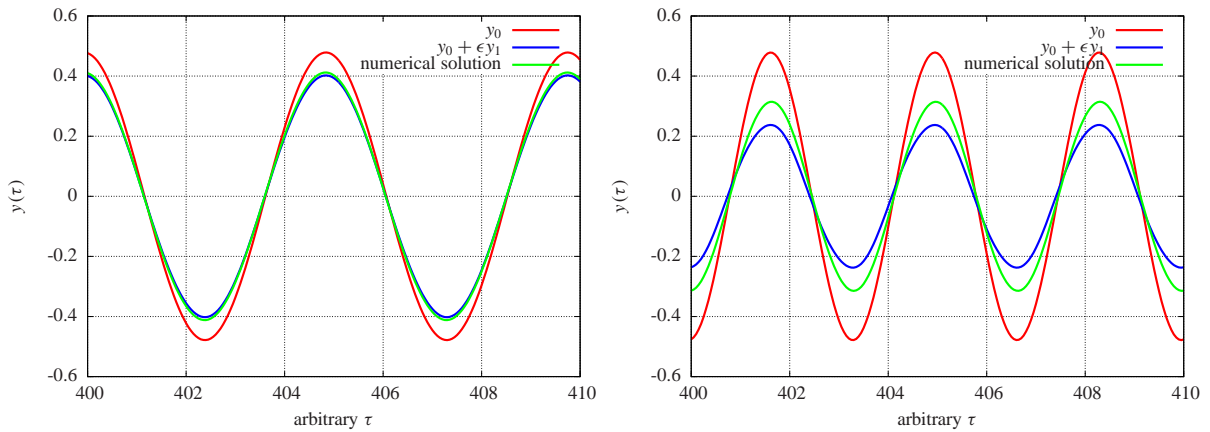


Figure 3. Comparison of solution y_0 and $y_0 + \varepsilon y_1$ for $r = 0.2$, $\Delta = 1$ and $\varepsilon = 0.28$ (left) and $\varepsilon = 0.88$ (right) with a fully numerical solution.

We note, however, that there is always the assumption that $\Delta = O(1)$ and $1 - \Omega^2 = O(\varepsilon)$. In other words, the validity of the resonance solution is for an interval in frequency of $\omega = \omega_0(1 + O(\varepsilon))$. When we leave this interval, the non-resonant solution (17) should gradually become applicable.

V. Impedance Calculation

In order to obtain realistic numbers, we will consider the impedance Z as the effective impedance of an array of Helmholtz resonators, where the spatially averaged neck velocity is identified to the external

acoustic velocity. Therefore, we add a porosity factor S_n/S_b to u_n and obtain

$$v_{\text{ex}} = \frac{S_n}{S_b} u_n. \quad (39)$$

Then we have to define what we mean with impedance for a sound field that is not entirely harmonic anymore. The natural choice is to define the impedance as the ratio of the Fourier transforms of the external pressure p_{ex} and (minus) the external velocity v_{ex} at excitation frequency ω . Taking the Fourier transforms of p_{ex} and v_{ex} , we have for $\eta > 0$

$$\hat{p}_{\text{ex}}(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p_{\text{ex}}(t) e^{-i\eta t} dt = \varepsilon^2 \rho_0 c_0^2 \frac{\ell S_n}{V} F_0 \delta(\eta - \omega) \quad (40)$$

and

$$\begin{aligned} \hat{v}_{\text{ex}}(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v_{\text{ex}}(t) e^{-i\eta t} dt = \frac{S_n}{S_b} \varepsilon \omega \ell e^{-i\theta} \left[-i(A_0 + \varepsilon A_1) \delta(\eta - \omega) \right. \\ \left. + \frac{2}{9\pi} \varepsilon A_0 |A_0| \delta(\eta - \omega) - \frac{1}{4\pi} \varepsilon A_0 |A_0| \sum_{n=1}^{\infty} \frac{(2n+1) \delta(\eta - (2n+1)\omega)}{n(n+1)(n^2 - \frac{1}{4})(n + \frac{3}{2})} + \dots \right], \end{aligned} \quad (41)$$

and so (with $V = LS_b$) we obtain³

$$Z(\omega) = -\frac{\hat{p}_{\text{ex}}(\omega)}{\hat{v}_{\text{ex}}(\omega)} = \frac{\varepsilon \rho_0 c_0^2 F_0}{L\omega} \cdot \frac{-i e^{i\theta}}{A_0 + \varepsilon A_1 + i \frac{2}{9\pi} \varepsilon A_0 |A_0|}. \quad (42)$$

It is interesting to consider Z to leading order in ε

$$Z(\omega) \simeq \frac{\varepsilon \rho_0 c_0^2 F_0}{L\omega} \frac{-i e^{i\theta_0}}{A_0} = \rho_0 c_0 \frac{c_0}{\omega L} \left(\frac{R}{\rho_0 \omega_0 \ell} + \frac{4}{3\pi} \frac{\|u_n\|}{\omega \ell} + 2i \frac{\omega - \omega_0}{\omega_0} \right) \quad (43)$$

(where $\|u_n\|$ denotes the amplitude of u_n) and observe that indeed $\text{Re}(Z)$ is of the often assumed form $a + b\|u_n\|$. Although our a and b are no constants and depend on ω , this is a higher order effect because $\omega = \omega_0(1 + O(\varepsilon))$. To leading order they are constant. $\text{Im}(Z)$ is independent of the excitation amplitude.

In order to illustrate formula (42), we have plotted in Fig. 4 resistance $\text{Re}(Z)$ and reactance $\text{Im}(Z)$ as a function of Ω , obtained for a typical geometry at different driving amplitudes, corresponding with ε varying from 0.05 to 0.28. As may be expected from (43), the main effect of the forcing amplitude is in the resistance. The reactance is practically independent of it. Typically, the resistance, being highest at or near the resonance frequency and decaying along both sides, increases everywhere with the amplitude, but more for frequencies less than resonance.

This behaviour may be compared in Fig. 5 with the measurements and predictions given by Motsinger and Kraft in [11]. Their predictions are (a.o.) based on a resistance of the form $R = \rho_0 c_0 (a + b|v|)$ with suitably chosen a and b . Unfortunately, only little experimental data for the higher amplitudes are available. The parameter values we used are based on $\omega_0/2\pi = 2200$ Hz, $\ell = 0.001$ m, $L = 0.035$ m, $S_n/S_b = 0.05$, $r = 0.1$, and ε -values for 125 dB corresponding with $\varepsilon = 0.3125$, 130 dB with $\varepsilon = 0.4168$, 135 dB with $\varepsilon = 0.5558$, 140 dB with $\varepsilon = 0.7411$ and 145 dB with $\varepsilon = 0.9883$.

The agreement is reasonable, taking into account that the ε 's are not very small and no experimental data are available in this frequency range for the higher amplitudes. Especially the increase of the maximum with the amplitude is confirmed. Only for the highest amplitude (with a value of $\varepsilon = 0.99$ that is far beyond what could be considered asymptotically "small") and frequencies well above resonance the decay predicted by [11] is not confirmed.

³For ω well away from resonance, we may obtain from (17) in a similar way the usual

$$Z(\omega) \simeq \frac{S_b}{S_n} \left(R + i \rho_0 \ell \omega_0 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right).$$

Being void of nonlinear effects, this case will not be considered here further.

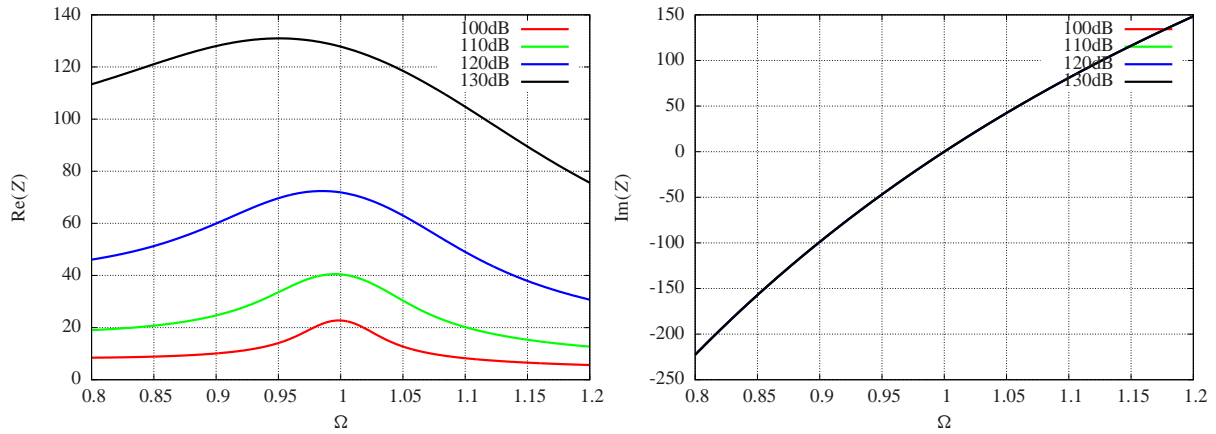


Figure 4. Real and imaginary parts of impedance Z for a Helmholtz resonator as a function of nondimensional frequency at different driving amplitudes. The realistic configuration that is chosen corresponds with $S_n/S_b = 0.05$, $r = 0.2$, $\omega_0 = 9086.9$ rad/s, $L = 0.035$ m, $\ell = 0.002$ m. 100 dB corresponds with $\varepsilon = 0.0497$, 110 dB with $\varepsilon = 0.0884$, 120 dB with $\varepsilon = 0.1572$, and 130 dB with $\varepsilon = 0.2796$.

V.A. Comparison of Impedances Based on y_0 and $y_0 + \varepsilon y_1$ Approximations

It is of interest to know when the driving amplitude becomes large enough to warrant the extra term εy_1 in the approximation of Z . Shown in Fig. 6 is the comparison of the impedance values obtained from y_0 and $y_0 + \varepsilon y_1$ approximations for different values of ε . Taking the same realistic geometry as above (Fig. 4), the value of ε vary from ~ 0.05 to 0.28 as the external driving amplitude is changed from 100 dB to 130 dB. We see that $O(\varepsilon)$ correction in the resulting resistance (the reactance is practically independent, especially near resonance) can be neglected for the lower amplitudes, but is indeed essential for the higher amplitudes.

VI. Conclusions

A systematic approximation of the hydrodynamically non-linear Helmholtz resonator equation is obtained, including the resulting impedance if the resonator is applied in an acoustic liner. To leading order, the usually assumed form of the resistance, $a + b|v|$, is recovered.

Our approach, based on systematic use of asymptotic analysis, allows higher order corrections, which indeed are shown to be important and relevant for practical configurations involving high amplitudes.

The real part of the found impedance (the resistance) shows the usual characteristic behaviour as a function of frequency, namely a maximum at or near the resonance frequency and a decay along both sides. All values increase with the amplitude, but slightly more for the frequencies less than resonance.

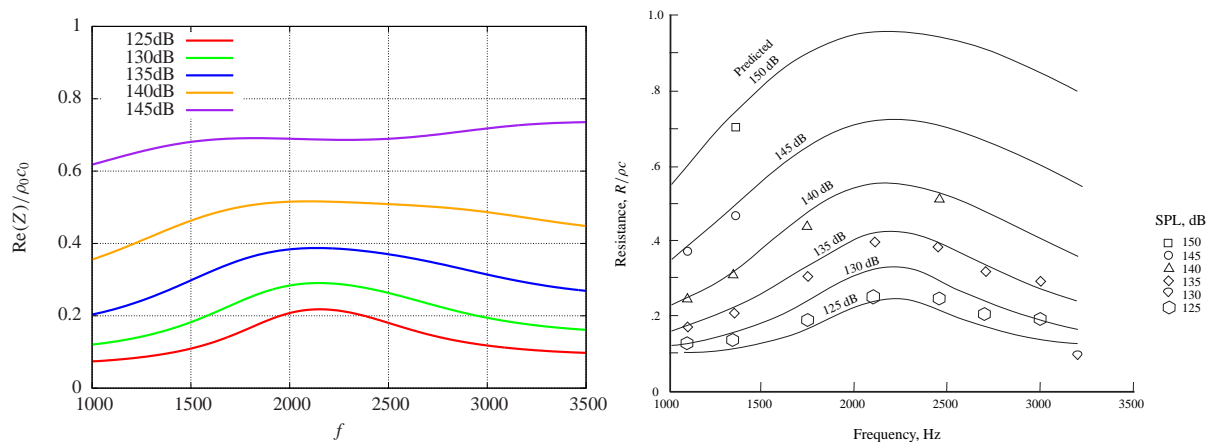


Figure 5. Comparison of (42) with measurements and predictions of $\text{Re}(Z)/\rho_0 c_0$ given in [11]. Estimated values of ε vary from 0.31 to 0.99.

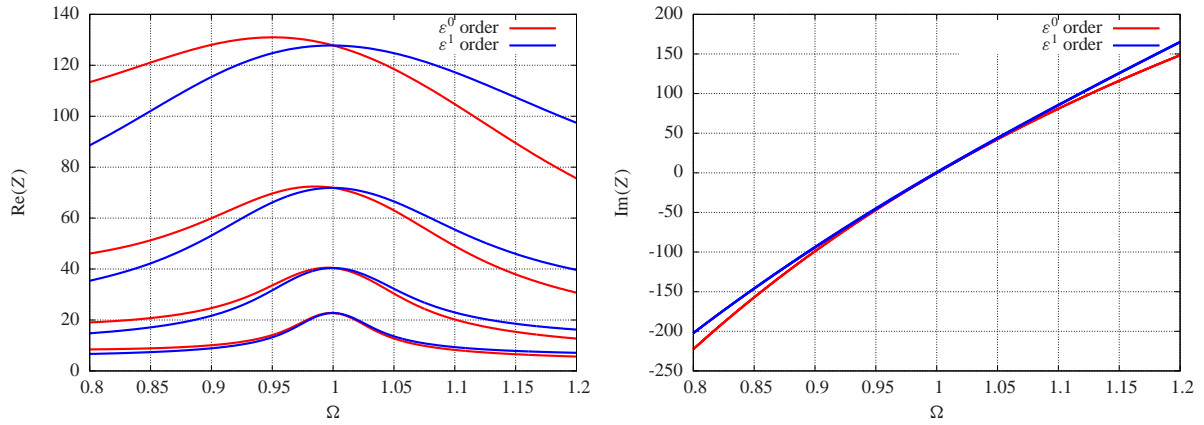


Figure 6. Comparison of the impedance obtained from the y_0 and $y_0 + \varepsilon y_1$ approximations for 100 dB, 110 dB, 120 dB and 130 dB. Same configuration as in Fig. 4

The imaginary part of the impedance (the reactance) is linear in frequency in a way that it vanishes at resonance and is practically independent of the amplitude.

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Appendix. Stability of Stationary Solution

From the physical origin of the problem, it is very likely that there exists a stable steady solution for a steady external forcing, such that we are not approximating a solution that just would not exist in any realisation. We have checked this mathematically by proving the boundedness of a small perturbation ξ of our solution y in (10), satisfying the following equation

$$(y'' + \xi'') + \varepsilon(y' + \xi')|y' + \xi'| + \varepsilon r(y' + \xi') + (y + \xi) = \varepsilon F(\tau). \quad (44)$$

Since, by assumption, y is any solution of the original equation, we have to linear order for small ξ (and a slight error near the zero's of $y' + \xi'$)

$$\xi'' + \phi(\tau)\xi' + \xi = 0 \quad (45)$$

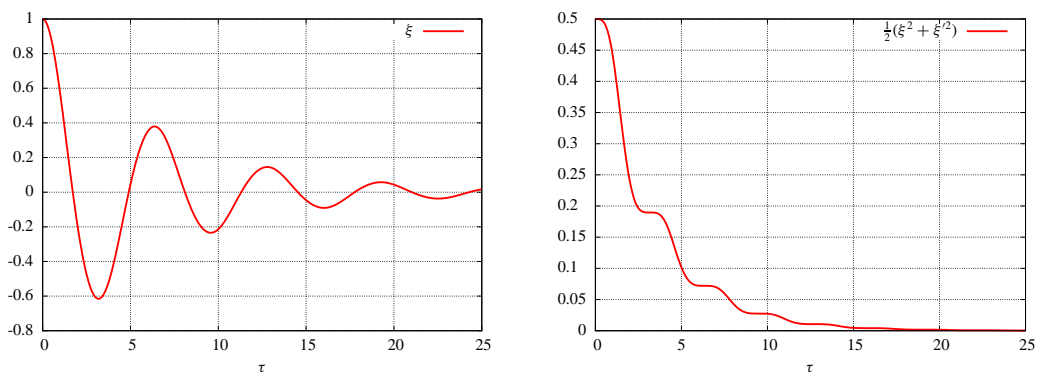


Figure 7. Amplitude ξ and mechanical energy $\frac{1}{2}(\xi^2 + \xi'^2)$ of a superimposed perturbation when it starts from $\xi(0) = 1$ $\xi'(0) = 0$. The full solution was approximated by $y = A_0 \cos(\Omega\tau)$ with $\varepsilon = 0.16$, $\Omega = 1$ and $r = 0.2$.

where $\phi(\tau) = \varepsilon(r + 2|y'|) \geq 0$ (even strictly positive if $r > 0$). We assume an initial condition with $\xi(0)^2 + \xi'(0)^2 = E_0^2$. From (45) we have for the mechanical energy $\frac{1}{2}(\xi^2 + \xi'^2)$

$$\frac{d}{d\tau} \left(\frac{1}{2}(\xi^2 + \xi'^2) \right) = \xi'\xi'' + \xi'\xi = -\phi\xi'^2 \leq -\phi(\xi^2 + \xi'^2).$$

It follows that $\frac{d}{d\tau} \ln(\xi^2 + \xi'^2) \leq -2\phi$. After integration and using the positivity of ϕ we find eventually

$$\xi^2 + \xi'^2 \leq E_0^2 \exp \left(-2 \int_0^\tau \phi(\tau) d\tau \right).$$

Hence it follows that perturbations are bounded and will decay to zero, confirming the existence of a stable stationary solution. Shown in Fig. 7 is a plot of ξ and $\frac{1}{2}(\xi^2 + \xi'^2)$.

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