

SOUND TRANSMISSION IN SLOWLY VARYING CIRCULAR AND ANNULAR LINED DUCTS WITH FLOW

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ABSTRACT

Sound transmission through straight circular ducts with a uniform inviscid mean flow and a constant acoustic lining (impedance wall) is classically described by a modal expansion. A natural extension for ducts with axially slowly varying properties (diameter and mean flow, wall impedance) is a multiple-scales solution. It is shown in the present paper that a consistent approximation of boundary condition and isentropic mean flow allows the multiple-scales problem to have an EXACT solution. Since the calculational complexities are no more than for the classical straight duct model, the present solution provides an attractive alternative to a full numerical solution if diameter variation is relevant. A unique feature of the present solution is that it provides a systematic approximation to the hollow-to-annular cylinder transition problem in the turbofan engine inlet duct.

1. INTRODUCTION

The theory of sound propagation in straight ducts with constant impedance type boundary conditions and a homogeneous stationary medium is classical and well-established (Morse and Ingard [7]; Pierce [14]). Per frequency ω , the sound field, satisfying Helmholtz' equation $(\nabla^2 + k^2)\phi = 0$, may be built up by superposition of eigensolutions or modes. These are certain shape-preserving fundamental solutions. The existence of these modes is a consequence of the relatively simple geometry, allowing separation of variables.

For cylindrical ducts, with associated cylindrical coordinate system (x, r, θ) the modes are given, in the usual complex notation, by exponentials and Bessel functions: $NJ_m(\alpha r)e^{i\omega t - im\theta - ikx}$ for a simple cylinder, and $[NJ_m(\alpha r) + MY_m(\alpha r)]e^{i\omega t - im\theta - ikx}$ for an annular cylinder.

The eigenvalue m , or circumferential wave number, is, due to the periodicity in θ , an integer; the eigenvalue α , or radial wave number, is determined by the appropriate boundary condition at the duct wall(s), while the axial wave number k is related to α and ω via

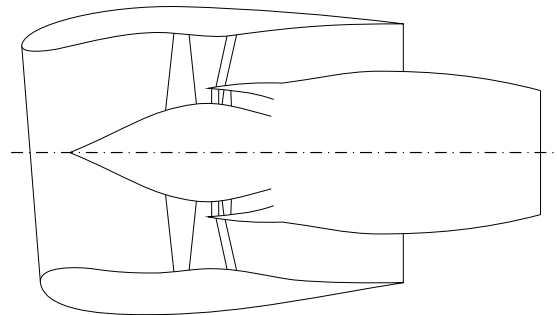


Figure 1: A turbofan engine

a dispersion relation. If we introduce a mean flow in the duct (motivated by aircraft turbofan engine applications, Nayfeh et al.[10], figure 1), the acoustic problem becomes rapidly much more difficult. Spatially varying mean flow velocities produce non-constant coefficients of the acoustic equations, which usually spoils the possibility of a modal expansion. Perhaps the simplest non-trivial mean flow is a uniform flow, in the limit of vanishing viscosity. Then modal solutions are possible, of a form rather similar to the one without flow.

A most important problem here is the way the sound field is transmitted through the vanishing mean flow boundary layer at the wall, which thus effectively modifies the impedance

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boundary condition *at* the duct wall into an equivalent boundary condition *in the limit* to the duct wall. This modified boundary condition was first proposed by Ingard [5], and later on proved by Eversman and Beckemyer [2] and Tester [17] to be indeed the correct limit for a boundary layer which is much smaller than a typical acoustic wave length.

In certain applications the geometry of a cylindrical duct is only an approximate model, and it is therefore of practical interest to consider sound transmission through ducts of varying cross section. In general, this problem is, again, very difficult, and one usually resorts to numerical methods. However, quite often, especially when the duct carries a mean flow, the diameter variations of the duct are only gradual thus introducing prospects of perturbation solutions. Indeed, several authors have utilized the small parameter related to the slow cross section variations (Eisenberg and Kao [1]; Tam [16]; Huerre and Karamcheti [4]; Thompson and Sen [18]). A particularly interesting and systematic approach is the method of multiple scales elaborated by Nayfeh and co-workers, both for ducts without (Nayfeh and Telionis [12]) and with flow (Nayfeh, Telionis and Lekoudis [13]; Nayfeh et al.[11]), and with hard and impedance walls. The multiple-scales technique provides a very natural generalization of modal solutions since a mode of a constant duct is now assumed to vary its shape according to the duct variations, in a way that amplitude and wave numbers are slowly varying functions, rather than constants.

In Rienstra [20] we proceeded along these lines, and presented an explicit multiple-scales solution of a problem similar to the one considered previously by Nayfeh et al. We considered a mode propagating in a slowly varying duct with impedance walls and containing almost uniform (inviscid, isentropic, irrotational) mean flow with vanishing boundary layer.

A somewhat puzzling aspect of Nayfeh et al.'s solutions was that without flow the differential equation for the slowly varying amplitude could be solved exactly, whereas with flow this was not the case. Also, in Rienstra [15] the amplitude equation for a similar problem of a duct with (slowly varying) porous walls could be solved exactly. In [20] we showed that, at least in the present type of problems, an exact solution appears to be the rule rather than an exception, *if* the entire perturbation analy-

sis is consistent at all levels. In the problem under consideration, Nayfeh et al. used an ad hoc mean flow velocity profile (quasi one dimensional with some assumed boundary layer) which is not a solution of the mean flow equations, and, furthermore, in case of a vanishing boundary layer they used an incorrect effective boundary condition, although at that time this was not known. Myers [8] showed that Ingard's [5] effective boundary condition for an impedance wall with uniform mean flow is to be modified significantly in case of non-uniform mean flow along curved surfaces.

Both Myers' [8] boundary condition and a consistent approximation of the mean flow is essential for the explicit solution presented.

In the present study we continued along these lines, and extended the theory to include an annular cylindrical geometry, in particular the transition from hollow to annular cylinder, and included some illustrative examples. These examples are taken from turbofan engine applications.

2. FORMULATION OF THE PROBLEM

We consider a cylindrical duct with slowly varying cross section. Inside this duct we have a compressible inviscid perfect isentropic irrotational gas flow, consisting of a mean flow and acoustic perturbations. To the mean flow the duct is hard-walled, but for the acoustic field the duct is lined with an impedance wall.

It is convenient to make dimensionless: spatial dimensions on a typical duct radius R_∞ , densities on a reference value ρ_∞ , velocities on a reference sound speed c_∞ , time on R_∞/c_∞ , pressure on $\rho_\infty c_\infty^2$, and velocity potential on $R_\infty c_\infty$. Note that the corresponding reference pressure p_∞ satisfies $\rho_\infty c_\infty^2 = \gamma p_\infty$.

We then have in the cylindrical coordinates (x, r, θ) , with unit vectors \mathbf{e}_x , \mathbf{e}_r and \mathbf{e}_θ , the duct inner wall radius R_1 and outer wall radius R_2 given by

$$r = R_1(X), \quad r = R_2(X), \quad X = \varepsilon x, \\ -\infty < x < \infty, \quad 0 \leq \theta < 2\pi,$$

where ε is a small parameter, and $R_{1,2}$ is by assumption only dependent on ε through εx . As we will see is ε absent in the final results, and its rôle is only to legitimize and support the present systematic perturbation method. The

fluid in the duct is described by (see, for example, Pierce [14]).

$$\begin{aligned}\tilde{\rho}_t + \nabla \cdot (\tilde{\rho} \tilde{\mathbf{v}}) &= 0, \\ \tilde{\rho}(\tilde{\mathbf{v}}_t + \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}) + \nabla \tilde{p} &= 0, \\ \gamma \tilde{p} &= \tilde{\rho}^\gamma, \quad \tilde{c}^2 = \frac{d\tilde{p}}{d\tilde{\rho}} = \tilde{\rho}^{\gamma-1}\end{aligned}\quad (2.1)$$

(with boundary and initial conditions), where $\tilde{\mathbf{v}}$ is particle velocity, $\tilde{\rho}$ is density, \tilde{p} is pressure, \tilde{c} is sound speed, and γ is the specific heat ratio, a constant. Since we assumed the flow to be irrotational, we may introduce a velocity potential $\tilde{\phi}$, such that $\tilde{\mathbf{v}} = \nabla \tilde{\phi}$. Using the vector identity $(\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} = \frac{1}{2} \nabla |\tilde{\mathbf{v}}|^2 + (\nabla \times \tilde{\mathbf{v}}) \times \tilde{\mathbf{v}} = \frac{1}{2} \nabla |\tilde{\mathbf{v}}|^2$, and the relation between \tilde{p} and $\tilde{\rho}$, the above momentum equation may be integrated to a variant of Bernoulli's equation

$$\frac{\partial \tilde{\phi}}{\partial t} + \frac{1}{2} |\tilde{\mathbf{v}}|^2 + \frac{c^2}{\gamma - 1} = \text{a constant} \quad (2.2)$$

This flow is split up into a stationary (mean) flow part, and an acoustic perturbation. This acoustic part varies harmonically in time with circular frequency ω , and with small amplitude to allow linearization. To avoid a complicating coupling between the two small parameters (ε and the acoustic amplitude), we assume this acoustic part much smaller than any relevant power of ε . In the usual complex notation (where the real part is assumed) we write then

$$\begin{aligned}\tilde{\mathbf{v}} &= \mathbf{V} + \mathbf{v} e^{i\omega t}, \quad \tilde{\phi} = \Phi + \phi e^{i\omega t}, \\ \tilde{\rho} &= D + \rho e^{i\omega t}, \quad \tilde{p} = P + p e^{i\omega t}, \\ \tilde{c} &= C + c e^{i\omega t}.\end{aligned}$$

Substitution and linearization yields:

- mean flow field

$$\begin{aligned}\nabla \cdot (D\mathbf{V}) &= 0 \\ \frac{1}{2} |\mathbf{V}|^2 + \frac{C^2}{\gamma - 1} &= E, \quad \text{a constant}, \\ \gamma P &= D^\gamma, \quad C^2 = \gamma P/D = D^{\gamma-1},\end{aligned}\quad (2.3)$$

- acoustic field

$$\begin{aligned}i\omega \rho + \nabla \cdot (D\nabla \phi + \rho \mathbf{V}) &= 0, \\ i\omega \phi + \mathbf{V} \cdot \nabla \phi + \frac{p}{D} &= 0, \\ p = C^2 \rho, \quad c &= \frac{1}{2} (\gamma - 1) D^{-\frac{1}{2}(\gamma+1)} p.\end{aligned}\quad (2.4)$$

The integration constant in the integrated momentum equation may be absorbed by ϕ . For the mean flow the duct wall is solid, so the normal velocity vanishes

$$\mathbf{V} \cdot \mathbf{n}_i = 0 \quad \text{at } r = R_i(X) \quad (i = 1, 2) \quad (2.5)$$

where the outward directed normal vectors at the wall are given by

$$\mathbf{n}_1 = -\frac{\mathbf{e}_r - \varepsilon R'_1 \mathbf{e}_x}{(1 + \varepsilon^2 R_1'^2)^{\frac{1}{2}}}, \quad \mathbf{n}_2 = \frac{\mathbf{e}_r - \varepsilon R'_2 \mathbf{e}_x}{(1 + \varepsilon^2 R_2'^2)^{\frac{1}{2}}}.$$

To define the mean flow an axial mass flux πF will be assumed such that the flow is subsonic everywhere. For the acoustic part the duct walls are locally reacting impedance walls with complex impedances $Z_1 = Z_1(X)$ and $Z_2 = Z_2(X)$ —slow variations of Z_i in X may be included—, meaning that at the wall, at a hypothetical point with zero mean flow,

$$p = Z_i(\mathbf{v} \cdot \mathbf{n}_i).$$

However, this is not the boundary condition needed here. Since we deal with a fluid of vanishing viscosity, the boundary layer along the wall in which the mean flow tends to zero is of vanishing thickness, and we cannot apply a boundary condition at the wall. The required condition is for a point near the wall but still (just) inside the mean flow. For arbitrary mean flow along a (smoothly) curved wall it was given by Myers [8] (eq. 15):

$$\begin{aligned}i\omega(\mathbf{v} \cdot \mathbf{n}_i) &= \\ \left[i\omega + \mathbf{V} \cdot \nabla - \mathbf{n}_i \cdot (\mathbf{n}_i \cdot \nabla \mathbf{V}) \right] \left(\frac{p}{Z_i} \right) & \\ \text{at } r = R_i(X) \quad (i = 1, 2) &\end{aligned}\quad (2.6)$$

with the remark that for simplicity we will exclude here the case $Z_i = 0$. Moreover, in terms of our small parameter ε we will assume $Z_i = O(1)$. The above equations and boundary conditions are evidently still insufficient to define a unique solution, and we need additional conditions for mean flow and sound field. This will be done by *assuming* a certain behaviour. Since we are studying axial variations due to the geometry of the pipe, the natural choice is to consider a mean flow, almost uniform with axial variations only in X , and a sound field consisting of a constant-duct mode perturbed by the X -variations. Furthermore, this choice indeed implies the absence of vorticity (apart from the vortex sheet along the wall), allowing the introduction of a velocity potential.

Before turning to the acoustic problem, we will derive in the next section the solution of the mean flow problem as a series expansion in ε . As noted before, a consistent mean flow expansion is necessary to obtain the explicit multiple scale solution of the acoustic problem.

3. MEAN FLOW

Since we assumed a mean flow, nearly uniform with axial variations in X only, we have

$$\mathbf{V} = U(X, r; \varepsilon)\mathbf{e}_x + V(X, r; \varepsilon)\mathbf{e}_r.$$

The cross-sectional mass flux is given by

$$2\pi \int_{R_1(X)}^{R_2(X)} D(X, r; \varepsilon)U(X, r; \varepsilon)r \, dr = \pi F, \quad \text{a constant.} \quad (3.1)$$

Since the variations in x are through X only, we may assume the constants E and F to be independent of ε . Furthermore, by writing out the same mass equation (2.3a) in X and r , it follows that the small axial mass variations can only be balanced by small radial variations, so $V = O(\varepsilon)$, and hence

$$U(X, r; \varepsilon) = U_0(X) + O(\varepsilon^2), \\ V(X, r; \varepsilon) = \varepsilon V_1(X, r) + O(\varepsilon^3),$$

and so, with equations (2.3b) and (2.3c),

$$P(X, r; \varepsilon) = P_0(X) + O(\varepsilon^2), \\ D(X, r; \varepsilon) = D_0(X) + O(\varepsilon^2), \\ C(X, r; \varepsilon) = C_0(X) + O(\varepsilon^2).$$

From equation (3.1) it follows now immediately that

$$U_0(X) = \frac{F}{D_0(X)(R_2^2(X) - R_1^2(X))} \quad (3.2)$$

with D_0 , P_0 and C_0 given by

$$\frac{1}{2} \left(\frac{F}{D_0(R_2^2 - R_1^2)} \right)^2 + \frac{1}{\gamma - 1} D_0^{\gamma-1} = E, \quad (3.3) \\ P_0 = \frac{1}{\gamma} D_0^\gamma, \quad C_0 = D_0^{\frac{\gamma-1}{2}},$$

where D_0 is to be determined numerically, per X . For V_1 , we return to the continuity equation, which is to leading order

$$\frac{\partial}{\partial X}(D_0 U_0) + \frac{1}{r} \frac{\partial}{\partial r}(r D_0 V_1) = 0.$$

Under the boundary conditions

$$-\frac{dR_i}{dX} U_0 + V_1 = 0 \\ \text{at } r = R_i(X) \quad (i = 1, 2),$$

(one of which is already satisfied through the application of (3.1) leading to (3.2)), we obtain the solution

$$V_1(X, r) = -\frac{F}{2r D_0(X)} \frac{\partial}{\partial X} \left(\frac{r^2 - R_1^2(X)}{R_2^2(X) - R_1^2(X)} \right) \quad (3.4)$$

The above solutions U_0 , P_0 , D_0 may be recognized as the well-known one dimensional gas flow equations (e.g., Liepmann et al. [6]). It should be stressed, however, that the radial velocity component V_1 is essential for a consistent mean flow description, and therefore necessary here.

4. THE ACOUSTIC FIELD

In this section we will derive the main result of the present paper: the explicit multiple-scales solution for a mode-like wave described by equation (2.4) with (2.6). When we eliminate p and ρ we have the following differential equation and boundary conditions for ϕ .

$$\nabla \cdot (D \nabla \phi) - D(i\omega + \mathbf{V} \cdot \nabla) \dots \\ \dots \left[\frac{1}{C^2} (i\omega + \mathbf{V} \cdot \nabla) \phi \right] = 0, \quad (4.1) \\ i\omega(\nabla \phi \cdot \mathbf{n}_i) = - \left(i\omega + \mathbf{V} \cdot \nabla - \mathbf{n}_i \cdot (\mathbf{n}_i \cdot \nabla \mathbf{V}) \right) \dots \\ \dots \left[\frac{D}{Z_i} (i\omega + \mathbf{V} \cdot \nabla) \phi \right] \quad \text{at } r = R_i(X).$$

A straight-duct modal wave form would be a function of r multiplied by a complex exponential in θ and x . The mode-like wave we are looking for here is obtained by assuming the amplitude and axial and radial wave numbers to be slowly varying, i.e. depending on X (Nayfeh et al.[12]). So we assume

$$\phi(x, r, \theta; \varepsilon) = A(X, r; \varepsilon) \dots \\ \dots \exp \left(-im\theta - i\varepsilon^{-1} \int^X \mu(\xi) \, d\xi \right) \quad (4.2)$$

Then the partial derivatives to x become formally (suppressing the exponential)

$$\frac{\partial}{\partial x} = -i\mu(X) + \varepsilon \frac{\partial}{\partial X}, \\ \frac{\partial^2}{\partial x^2} = -\mu(X)^2 - i\varepsilon \frac{d\mu}{dX} - 2i\varepsilon\mu(X) \frac{\partial}{\partial X} \\ + \varepsilon^2 \frac{\partial^2}{\partial X^2}.$$

Substitution in (4.1a), and collecting like powers of ε yield up to order ε^2

$$D_0\mathcal{L}(A) = \frac{i\varepsilon}{A} \left\{ \frac{\partial}{\partial X} \left[\left(\frac{U_0\Omega}{C_0^2} + \mu \right) D_0A^2 \right] + \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{V_1\Omega}{C_0^2} D_0A^2 \right] \right\} \quad (4.3)$$

where

$$\Omega = \omega - \mu U_0,$$

and the operator \mathcal{L} is defined by

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\Omega^2}{C_0^2} - \mu^2 - \frac{m^2}{r^2}.$$

The boundary conditions (4.1b), up to order ε^2 , are now

$$i\omega \frac{\partial A}{\partial r} + \frac{\Omega^2 D_0 A}{Z_1} = \varepsilon \omega \mu \frac{dR_1}{dX} A + \frac{i\varepsilon}{A} \left[U_0 \frac{\partial}{\partial X} + V_1 \frac{\partial}{\partial r} - \frac{\partial V_1}{\partial r} \right] \left(\frac{\Omega D_0 A^2}{Z_1} \right) \quad (r = R_1), \quad (4.4)$$

$$i\omega \frac{\partial A}{\partial r} - \frac{\Omega^2 D_0 A}{Z_2} = \varepsilon \omega \mu \frac{dR_2}{dX} A - \frac{i\varepsilon}{A} \left[U_0 \frac{\partial}{\partial X} + V_1 \frac{\partial}{\partial r} - \frac{\partial V_1}{\partial r} \right] \left(\frac{\Omega D_0 A^2}{Z_2} \right) \quad (r = R_2).$$

Now assume

$$A(X, r; \varepsilon) = A_0(X, r) + \varepsilon A_1(X, r) + \dots,$$

then substitution into equation (4.3) yields to leading order $\mathcal{L}(A_0) = 0$, which is, up to a radial coordinate stretching, Bessel's equation in r , with X acting only as a parameter. The mode-like solution we are looking for is then

$$A_0(X, r) = N(X)J_m(\alpha(X)r) + M(X)Y_m(\alpha(X)r)$$

where J_m and Y_m are the m -th order Bessel function of the first and second kind (Watson [19]). The reduced boundary conditions (4.4) produce the following equation for 'eigenvalue' α (continuous in X)

$$\frac{\alpha R_2 J'_m(\alpha R_2) - \zeta_2 J_m(\alpha R_2)}{\alpha R_2 Y'_m(\alpha R_2) - \zeta_2 Y_m(\alpha R_2)} = \frac{\alpha R_1 J'_m(\alpha R_1) + \zeta_1 J_m(\alpha R_1)}{\alpha R_1 Y'_m(\alpha R_1) + \zeta_1 Y_m(\alpha R_1)} = -\frac{M(X)}{N(X)}, \quad (4.5)$$

where

$$\zeta_1 = \frac{\Omega^2 D_0 R_1}{i\omega Z_1}, \quad \zeta_2 = \frac{\Omega^2 D_0 R_2}{i\omega Z_2}.$$

Expression (4.5) itself is equal to $-M(X)/N(X)$, so only N is to be determined. α and μ are related by the dispersion relation

$$\alpha^2 + \mu^2 = \Omega^2 / C_0^2.$$

It is convenient to introduce the reduced axial wave number

$$\sigma = \sqrt{1 - (C_0^2 - U_0^2) \frac{\alpha^2}{\omega^2}}$$

so that

$$\mu = \omega \frac{C_0\sigma - U_0}{C_0^2 - U_0^2}, \quad \frac{U_0\Omega}{C_0^2} + \mu = \frac{\omega\sigma}{C_0},$$

$$\Omega = \omega C_0 \frac{C_0 - U_0\sigma}{C_0^2 - U_0^2}.$$

The branch (i.e. sign) of σ is to be selected such, that $\text{Im } \sigma \leq 0$, $\text{Re } \sigma \geq 0$ (IV-th quadrant) if the mode is propagating in positive direction, and $\text{Im } \sigma \geq 0$, $\text{Re } \sigma \leq 0$ (II-nd quadrant) if the mode is propagating in negative direction. A single exception is to be made if impedance, frequency, and mean flow are such that the vortex sheet between mean flow and impedance wall becomes (Helmholtz) unstable, corresponding to a σ in either the I-st or III-d quadrant ([21],[22],[23] and others). Although it wouldn't alter the present results, we will not consider these cases here.

Note that in the cylindrical duct case, with $R_1 = 0$, we have just $M(X) = 0$ so that

$$A_0(X, r) = N(X)J_m(\alpha(X)r),$$

and α is determined from

$$\alpha R_2 J'_m(\alpha R_2) - \zeta_2 J_m(\alpha R_2) = 0. \quad (4.6)$$

The amplitude functions $N(X)$ and $M(X)$ are determined from the condition that there exists a solution A_1 . This is not trivial since we assumed the solution to behave in a certain way, namely, to depend on X rather than x . Now suppose that we would proceed and solve the equation for A_1 , and subsequently find the necessary forms of N and M , then it would appear that we end up with similarly undetermined functions in A_1 . So this approach looks rather inefficient. Indeed, it is not necessary to work out the equations for A_1 in detail. We only need a solvability condition (Nayfeh [9]), sufficient to yield the required equation for N .

Since the operator $r\mathcal{L}$ is self-adjoint in r , we have

$$\int_{R_1}^{R_2} A_0 \mathcal{L}(A_1) r \, dr = R_2 \left[A_0 \frac{\partial A_1}{\partial r} - A_1 \frac{\partial A_0}{\partial r} \right]_{r=R_2} - R_1 \left[A_0 \frac{\partial A_1}{\partial r} - A_1 \frac{\partial A_0}{\partial r} \right]_{r=R_1}.$$

Further evaluation of this expression (using (4.4)) and the corresponding right-hand side of (4.3) gives finally, after some calculation, the following equation

$$\frac{d}{dX} \left[D_0 \frac{\omega\sigma}{C_0} \int_{R_1}^{R_2} A_0^2(X, r) r \, dr + \frac{D_0 U_0}{\Omega} \left(\zeta_2 A_0^2(X, R_2) + \zeta_1 A_0^2(X, R_1) \right) \right] = 0. \quad (4.7)$$

Use is made of equations (3.2) and (3.5), and the identities

$$\int^{R(X)} \frac{\partial}{\partial X} f(X, r) \, dr = \frac{d}{dX} \int^{R(X)} f(X, r) \, dr - \frac{dR}{dX} f(X, R),$$

and

$$U_0 \frac{\partial}{\partial X} + V_1 \frac{\partial}{\partial r} = U_0 \frac{d}{dX} \text{ along } r = R(X).$$

The above equation (4.7) can be integrated immediately, with a constant of integration Q_0^2 . This constant is determined by the initial amplitude of the mode entering the duct. Since the solution is linear, it is further irrelevant here. The integral of $A_0^2 r$, finally to be evaluated, is a well-known integral of Bessel functions (Appendix), with the result (using (4.5))

$$\int_{R_1}^{R_2} A_0^2(X, r) r \, dr = \frac{1}{2} R_2^2 \left(1 - \frac{m^2 - \zeta_2^2}{\alpha^2 R_2^2} \right) A_0^2(X, R_2) - \frac{1}{2} R_1^2 \left(1 - \frac{m^2 - \zeta_1^2}{\alpha^2 R_1^2} \right) A_0^2(X, R_1).$$

Using the following expressions

$$A_0(X, R_1) = \frac{(2/\pi)N(X)}{\alpha R_1 Y'_m(\alpha R_1) + \zeta_1 Y_m(\alpha R_1)}$$

$$A_0(X, R_2) = \frac{(2/\pi)N(X)}{\alpha R_2 Y'_m(\alpha R_2) - \zeta_2 Y_m(\alpha R_2)}$$

and some further simplifications we thus obtain for $N(X)$

$$\left(\frac{\frac{1}{2}\pi Q_0}{N} \right)^2 = \frac{D_0 \omega \sigma R_2^2 \left(1 - \frac{m^2 - \zeta_2^2}{\alpha^2 R_2^2} \right) + \frac{D_0 U_0}{\Omega} \zeta_2}{\left(\alpha R_2 Y'_m(\alpha R_2) - \zeta_2 Y_m(\alpha R_2) \right)^2} \dots - \frac{D_0 \omega \sigma R_1^2 \left(1 - \frac{m^2 - \zeta_1^2}{\alpha^2 R_1^2} \right) - \frac{D_0 U_0}{\Omega} \zeta_1}{\left(\alpha R_1 Y'_m(\alpha R_1) + \zeta_1 Y_m(\alpha R_1) \right)^2}. \quad (4.8)$$

Expression (4.8) for $N(X)$ is the principal result of the present paper. An interesting special case is the hard-walled duct, where $Z_i = \infty$, $\zeta_i = 0$. Then we have

$$\left(\frac{\frac{1}{2}\pi Q_0}{N} \right)^2 = \frac{D_0 \omega \sigma}{2C_0} \left(\frac{R_2^2 - m^2/\alpha^2}{[\alpha R_2 Y'_m(\alpha R_2)]^2} - \frac{R_1^2 - m^2/\alpha^2}{[\alpha R_1 Y'_m(\alpha R_1)]^2} \right) \quad (4.9)$$

Note that α is real here. For a hollow cylinder, without inner wall, the above general result (4.8) reduces, in the limit $R_1 \rightarrow 0$, to

$$\left(\frac{Q_0}{N} \right)^2 = \left(\frac{D_0 \omega \sigma R_2^2}{2C_0} \left(1 - \frac{m^2 - \zeta_2^2}{\alpha^2 R_2^2} \right) + \frac{D_0 U_0}{\Omega} \zeta_2 \right) J_m(\alpha R_2)^2 \quad (4.10)$$

So the present solution is equally valid for hollow and annular cylindrical ducts, and hence includes the unique feature that it provides (apparently for the first time) a systematic approximation to the hollow-to-annular cylinder transition problem in the turbofan engine duct inlet. This aspect will be illustrated in the next section by an example.

If convenient, we may observe that for a hard walled hollow cylinder the combination αR_2 is a constant, independent of X , so we can absorb some constant factors into Q_0 to obtain

$$\left(\frac{Q_0}{N} \right)^2 = \frac{D_0 \omega \sigma R_2^2}{C_0}. \quad (4.11)$$

Of course, with a transition from a hollow to annular cylinder this is not advisable, because then it is required that we deal with the *same* Q_0 .

5. EXAMPLE

In this section we will discuss an example of the previous theory. A lined inlet duct of a CFM56-inspired turbofan engine, from inlet plane via (hard-walled) spinner to the inlet rotor plane, is given by $R_2, R_1, Z_2 = 2 - i$ and $Z_1 = \infty$ (figure (2)₁₁). Assuming the dimensionless density $D = 1$ far upstream, its value slightly below 1 at the inlet plane, and the inlet Mach number ~ 0.6 , we choose $F = 0.6764$ and $E = 2.5136$. Density and Mach number are given in figure (2)₁₂. A rotor blade number of 26, and a rotor tip Mach number slightly below 1 is taken, such that the first harmonic has frequency $\omega = 25$ and $m = 26$.

The first radial left running mode is considered, together (for comparison) with its right running companion. In figures (2)_{3,1-3} the axial wave number μ , radial wave number α , and reduced axial wave number σ are shown in the complex plane, parametrically varying with the duct position x . Initial positions are indicated by an open circle, intermediate positions by filled small circles.

To be sure that we are looking at the same left- and right-running mode, both are found first, at the initial x -position, for no-flow conditions ($F = 0$), when both modes coincide. Then the modes are traced for increasing F . This can be seen in figure (2)₃₂, the plot for α : the thin dotted line is α at $x = 0$ for increasing F .

The cross-sectionally averaged amplitude functions A , given by

$$\bar{A}(X) = \left[\int_{R_1}^{R_2} |A(X, r)|^2 r dr \right]^{1/2},$$

are plotted in figure (2)₂₁. The respective values are normalized to 1 at begin and end position.

Since it is of interest to measure the amount of dissipated acoustic energy, we introduce here the acoustic power \mathcal{P} of a single mode through a duct cross section. Following Goldstein [3], we define the acoustic power at a surface S

$$\mathcal{P} = \int_S \mathbf{I} \cdot \mathbf{n}_S ds,$$

where \mathbf{I} is the time-averaged acoustic intensity or energy flux, here given by

$$\mathbf{I} = \frac{1}{2} \text{Re}[(p/D + \nabla\Phi \cdot \nabla\phi)(D\nabla\phi + \rho\nabla\Phi)^*],$$

with $*$ denoting the complex conjugate. Considering here for S a duct cross section, we need the axial component of \mathbf{I} , which is to leading order

$$I_x = \frac{D_0\omega^2}{2C_0} \text{Re}(\sigma)|\phi|^2 = \frac{D_0\omega^2}{2C_0} \text{Re}(\sigma)|A_0|^2 \exp\left(2 \int^x \text{Im} \mu(\varepsilon\xi) d\xi\right). \quad (5.1)$$

so that

$$\mathcal{P} = 2\pi \frac{D_0\omega^2}{2C_0} \text{Re}(\sigma) \int_{R_1}^{R_2} |A_0(X, r)|^2 r dr \dots \dots \exp\left(2 \int^x \text{Im} \mu(\varepsilon\xi) d\xi\right), \quad (5.2)$$

where (see Appendix)

$$\int_{R_1}^{R_2} |A_0(X, r)|^2 r dr = \frac{|A_0(X, R_2)|^2 \text{Im} \zeta_2 + |A_0(X, R_1)|^2 \text{Im} \zeta_1}{\text{Im} \alpha^2},$$

which is, of course, equivalent to $(\bar{A})^2$. For hard-walled ducts ($Z_i = \infty, \zeta_i = 0$) all eigenvalues α are real. Then $\mathcal{P} = 0$ for cut-off modes ($\text{Re} \sigma = 0$). For cut-on modes, where σ is real, A_0 is also real and:

$$\int_{R_1}^{R_2} A_0(X, r)^2 r dr = \frac{Q_0^2 C_0}{D_0 \omega \sigma}.$$

Since the value of \mathcal{P} is highly dominated by the exponential part $e^{2 \int \text{Im} \mu d\xi}$, we have plotted the power both without (figure (2)₂₂, linear scale) and with this exponent (figure (2)₂₃, dB scale). Since the ‘‘head wind’’ moved the left running mode just into a cut-off position, this mode is much more damped, as could have been concluded from the behaviour of μ already.

6. DISCUSSION AND CONCLUSIONS

If the multiple-scales solution is valid, the mode-like wave behaves locally like a mode of a straight duct. Rotating with angular velocity ω/m , it propagates in axial direction with or without attenuation (unattenuated or cut on: $\text{Im} \mu = 0$; attenuated or cut off: $\text{Im} \mu \neq 0$). The more interesting aspects here are, of course, connected to the slow variations in X . These are mainly represented by the amplitude functions N, M and the phase function μ .

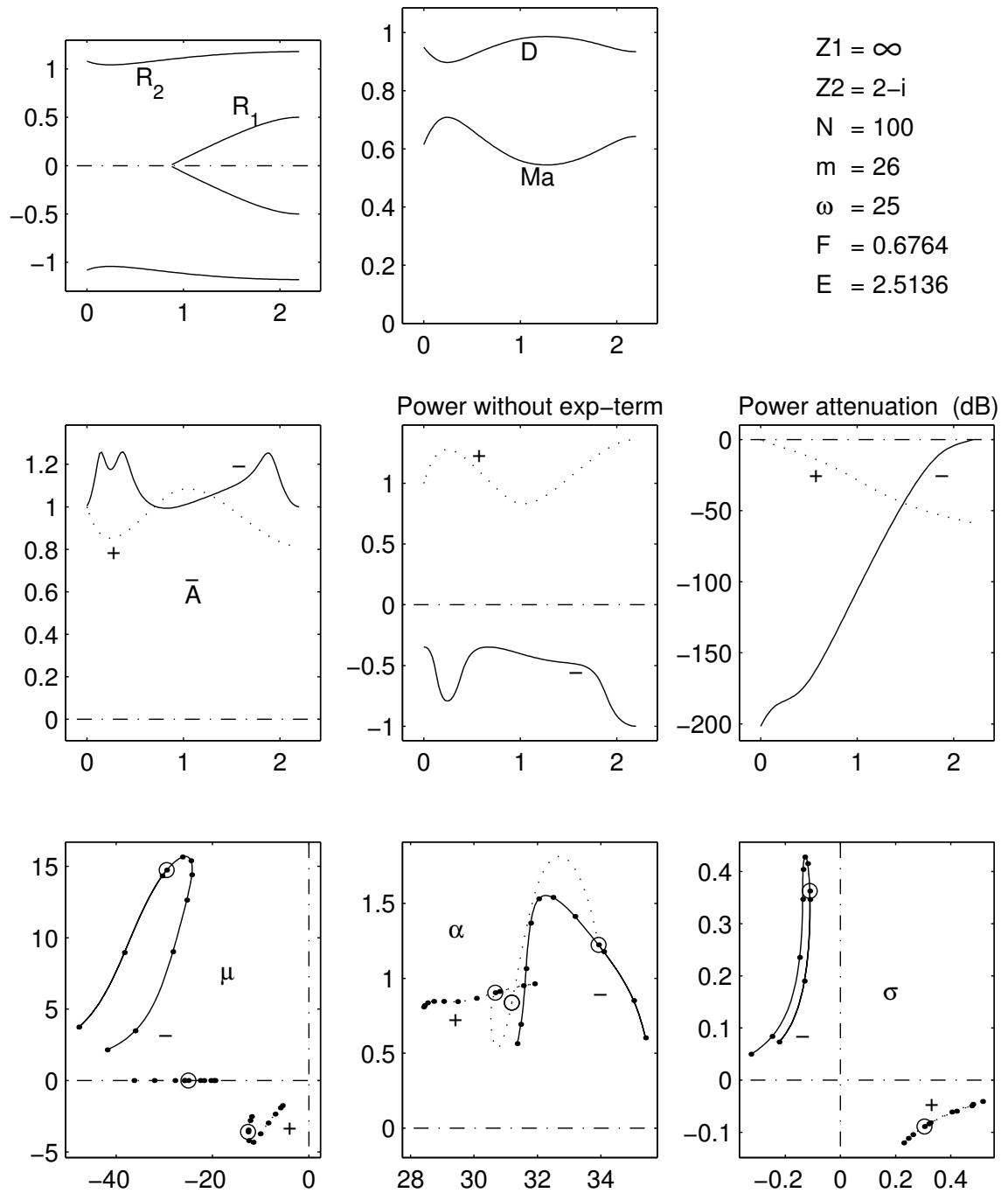


Figure 2: Sound propagation in the lined inlet duct of a turbofan engine.
 First radial (left- and right running) modes.

When R and Z vary with X , the mode changes gradually, except at the points where the denominator of N (eq. (4.8)) vanishes and the approximation breaks down. These points are just found at the double eigenvalues, i.e. where two eigenvalues μ (or α) coalesce. These are given by equation (4.5) and its partial derivative to μ .

Clearly, the approximation breaks down because the two coalescing modes couple (the energy of the incident mode is distributed over the two) in a short region. A local analysis, not given here, is necessary to determine the resulting amplitudes. In general the two modes propagate in the same direction but not necessarily. The second mode may run backwards while at the same time the incident mode becomes cut-off in such a way that beyond the point no energy is propagated. Points with this behaviour are usually called turning points, since the incident mode is totally reflected into the backward running mode (we assume, of course, the absence of tunnelling by other interfering turning points).

Turning points occur in practice with hard-walled ducts ($Z = \infty$), where a real σ tends to zero to become pure imaginary (α is always real). At $\sigma = 0$, N is singular (eq. (4.9, 4.11)), and the incident mode couples to a backwards running mode. For real σ we have

$$\mathcal{P} \sim \text{Re } \sigma \neq 0,$$

whereas for pure imaginary σ

$$\mathcal{P} = 0,$$

so the mode must reflect indeed. Note that this behaviour is irrespective of the presence of mean flow.

In conclusion, we have found an explicit solution for the multiple scale problem of modal sound propagation through slowly varying lined ducts with isentropic mean flow. It is shown that a consistent approximation of boundary condition and mean flow allows the multiple-scales problem to have an EXACT solution.

Since the calculational complexities are no more than for the classical straight duct model, the present solution provides an attractive alternative to a full numerical solution if diameter variation is relevant. The present solution is equally valid for hollow and annular cylindrical ducts, and hence includes the unique feature

that it provides a systematic approximation to the hollow-to-annular cylinder transition problem in the turbofan engine duct inlet. This aspect is elaborated by an example.

The solution remains equally valid for hard-walled or no-flow ducts, but needs adaptation for a completely soft wall with $Z = 0$. The approximation breaks down at double eigenvalues when the mode couples with other modes. This occurs for example at cut-off points in a hard-walled duct.

7. APPENDIX

Well-known integrals of Bessel functions (Watson [19], p.135) are

$$\begin{aligned} \int x \mathcal{C}_m(\alpha x) \tilde{\mathcal{C}}_m(\beta x) dx &= \frac{x}{\alpha^2 - \beta^2} \dots \\ &\left\{ \beta \mathcal{C}_m(\alpha x) \tilde{\mathcal{C}}'_m(\beta x) - \alpha \mathcal{C}'_m(\alpha x) \tilde{\mathcal{C}}_m(\beta x) \right\}, \\ \int x \mathcal{C}_m(\alpha x) \tilde{\mathcal{C}}_m(\alpha x) dx &= \\ &\frac{1}{2} \left(x^2 - \frac{m^2}{\alpha^2} \right) \mathcal{C}_m(\alpha x) \tilde{\mathcal{C}}_m(\alpha x) \\ &+ \frac{1}{2} x^2 \mathcal{C}'_m(\alpha x) \tilde{\mathcal{C}}_m(\alpha x), \end{aligned}$$

where \mathcal{C}_m and $\tilde{\mathcal{C}}_m$ is any linear combination of J_m and Y_m .

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