Sound Radiation from a Co-Planar and Buried Nozzle with Jet and Bypass Flow and Arbitrary Edge Condition

A. Demir & S.W. Rienstra

a.demir@tue.nl, s.w.rienstra@tue.nl

Department of Mathematics & Computer Science, Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

Abstract

In this report, the radiation of sound from an annular duct exit with buried inner nozzle system, with jet and bypass flow, is studied. This problem represents a simplified model of the radiation of fan and turbine noise from engine exhausts. The model consist of two semi-infinite ducts. The smaller duct (inner nozzle), whose exit plane coincides with the larger one or extends beyond it, is situated inside the larger semi-infinite duct. The inner nozzle issues the core flow inside the bypass jet flow. The bypass nozzle issues the bypass jet flow inside the ambient co-flow. Two vortex sheets, attached to the duct exit trailing edges, separate the different flows from each other. These vortex sheets are unstable due to this mean velocity discontinuity. The application of the Kutta condition at the respective trailing edges guarantees shedding of vorticity which excites these instabilities. The system is set up to respond to an incident annular duct mode, but the analysis is very similar for an inner duct mode.

Formulation of the boundary value problem, by following the classical approach, leads to a couple of simultaneous Wiener-Hopf equations. These equations produce a matrix equation system, which is formally decoupled by the introduction of an infinite sum of poles with coefficients to be determined. This method is known as Idemen's method of "weak factorisation" or "the pole removal technique". The uncoupled scalar equations are solved independently by a standard application of analytical continuation. The final solution includes unknown coefficients which are determined by solving an infinite linear algebraic system numerically. The contribution of the instability waves are separated from the rest of the solution.

The asymptotic far field is found by a standard application of the steepest descent method. Finally a series of practical examples are given.

1 Introduction

Turbomachinery noise from exhaust nozzles is an important part of the noise radiated from aircraft engines. Despite the progress made on the reduction of noise in recent years, there is still a lack of research on this complicated problem, where sound propagates through the shear layers of high speed hot jets. The European project TURNEX (co-ordinator Brian Tester) addresses this shortfall and aims to reduce aeroengine noise at source by developing enhanced numerical prediction techniques. To acquire reliable and competitive numerical solutions, exact solutions are needed for verification. This can be achieved by obtaining analytical solutions for simplified models of the problem.

The first step was taken by Munt [1] who developed a Wiener-Hopf solution [2] for the interaction of sound with a jet issuing from a hollow cylindrical duct with Kutta condition. This solution coincides with the classical solutions of Levine & Schwinger [3] for plane waves and no flow, higher order modes and no flow by Weinstein [4], plane waves with uniform mean flow by Carrier [5] and higher order modes in uniform mean flow by Homicz & Lordi [6], or the corresponding point source problem [7]. These last two articles derived their results by applying a Lorentz transform to obtain the no-flow solutions in potential form for the inlet and in pressure form for the exhaust case, which resulted in respectively a no-Kutta-condition solution and the Kutta-condition solution. The no-Kutta condition solution was the one that corresponds with no vortex shedding.

The role of the Kutta condition to determine the amount of vorticity, shed from the edge, in Munt's solution was not immediately seen to be related to the occurrence of the Kelvin-Helmholtz instability [8], but through the work of Bechert [9] and Howe [10] it was realised in [11] to be the same, and it was shown that the amount of vortex shedding is a parameter of the problem, such that no vortex shedding corresponds with no instability but also no Kutta condition, a certain amount of vortex shedding corresponds to excitation of the instability but not necessarily a smooth field near the edge, i.e. Kutta condition, and a unique amount of vortex shedding that corresponds to the Kutta condition and the practically maximum amplitude of instability.

After Munt's analytical exact solution a series of related configurations were developed: analytical approximations of Munt's solution for low Helmholtz number (Cargill [12]) and low Strouhal number (Rienstra [11, 13]); a uniform mean flow with annular duct [14] where a detailed acoustic energy bookkeeping could be given, similar to the 2D problem of [15]; the original Munt problem for annular geometry [16] was a natural next step; the annular geometry with full or partially lined center body [17] required an essentially different approach.

The doubly infinite centerbody obviously simplifies the solution for a finite but relatively long, protruding afterbody, but not for a centerbody exit that is co-planar with or even buried inside the outer duct, as can be found in certain turbofan engines [18]. As a part of the TURNEX project in this report we will assume that the semi-infinite centerbody is axially inserted into the outer semi-infinite duct. The dimensionless distance between the duct exit planes is defined by length l. The existence of a secondary exit (inner nozzle) turns the related boundary value problem into a matrix Wiener-Hopf equation which is more complex than the classical scalar one. By the usual Wiener-Hopf factorization methods such an equation does not seem to be solvable. However, for $l \leqslant 0$, it is possible to reduce the matrix Wiener-Hopf equation into a form to which the weak factorization (originally due to Idemen [19]; see also [20]) is applicable. Just like the classical Munt solution [1] and its variants the solution includes the effects of vortex shedding and Helmholtz instabilities of the inner and outer jet, while the mean flow is now triple piecewise constant in velocity, sound speed and density along the inner jet, the bypass jet and the outer flow.

The first who considered this same buried nozzle problem were Taylor, Crighton & Cargill [18]. However, they approximated the solution by assuming low Mach and Strouhal numbers and large l. In this report, we will consider the full problem without further approximations. The problem of a protruding inner exit can also be solved, but with greater effort, by approximating certain functions by Padé approximants, rendering them meromorphic in this way. This approach was followed by Veitch & Peake [21], parallel and independent to the present work.

In the first section the model is described and the large number of parameters of the problem are introduced. The linearized Euler equations in different regions are solved analytically using Fourier transforms. The boundary and continuity relations lead to expressions with unknown spectral Fourier coefficients and unknown semi-analytical functions. A couple of Wiener-Hopf equations including four unknown semi-analytic functions is constructed. Its solution, which involves coefficients determined by solving an infinite linear algebraic system numerically, is obtained with the help of the method of weak factorisation. Two constants which controls the amount of vortex shedding from inner and outer edge are included in the solution.

The solutions are of Fourier type integral form, which have to be calculated numerically. The second section deals with the far field approximation to these integrals with the well known method of steepest descent. For near field the necessary integrals must be calculated numerically.

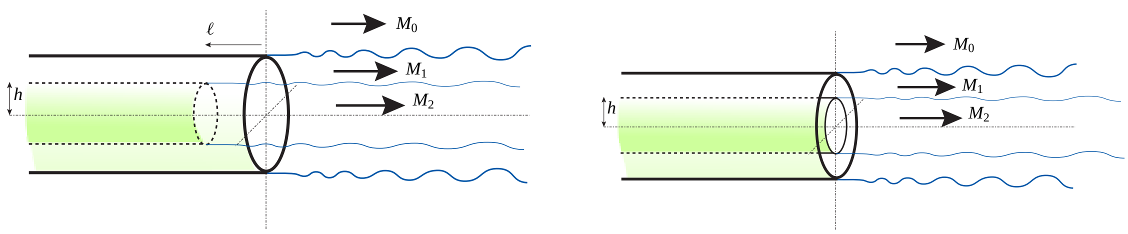
Because of the relatively expensive near field calculations, only far field results are presented in the numerical examples section. One combination of parameters are used for the calculations. Coplanar and buried nozzle models are chosen with taking $l = 0$ or $l < 0$, respectively. The results are obtained by a code developed at TUE, which is also a deliverable of TURNEX.

A restricted version of the present paper is presented at the ICSV14 congress as [22].

2 Analysis

2.1 Formulation of the Problem

Consider the geometry which consists of a semi-infinite outer duct and a semi-infinite inner duct. Duct walls are assumed to be infinitely thin and they occupy the regions like depicted in the figures 1(a), 1(b). All quantities are made dimensionless by using outer duct radius and ambient flow properties as reference as $r, z \sim R_2$, $U \sim c_0$, $\rho \sim \rho_0$, $t \sim R_2/c_0$. The velocity potential ϕ will be used to obtain



(a) Buried nozzle (i.e. $\ell < 0$)   (b) Coplanar nozzle (i.e. $\ell = 0$)

Figure 1: Geometry of annular duct with core flow nozzle

the acoustic pressure p, velocity $\mathbf{v}$ and density ρ via the following equations [17]

$$p = -(\phi_t + M_0\phi_z), \quad \mathbf{v} = \nabla\phi, \quad \rho = p, \quad 1 < r \tag{1a}$$
$$p = -D_1(\phi_t + M_1\phi_z), \quad \mathbf{v} = \nabla\phi, \quad \rho = pC_1^2, \quad h < r < 1 \tag{1b}$$
$$p = -D_2(\phi_t + M_2\phi_z), \quad \mathbf{v} = \nabla\phi, \quad \rho = pC_2^2, \quad 0 \leqslant r < h \tag{1c}$$

where $M_0 = U_0/c_0$, $M_1 = U_j/c_0$, $M_2 = U_c/c_0$, $C_1 = c_0/c_j$, $C_2 = c_0/c_c$, $D_1 = \rho_j/\rho_0$, $D_2 = \rho_c/\rho_0$. All mean flows are subsonic. Note that M_1 and M_2 are not the local Mach numbers.

From the symmetry of the geometry and the incident wave, the diffracted field will remain with the same azimuthal and time dependencies as the incident wave. The total field is written in different regions as:

$$\phi(r, z, \theta, t) = \psi^T(r, z)\exp(\mathrm{i}\omega t - \mathrm{i}m\theta) \tag{2a}$$
$$\psi^T(r, z) = \begin{cases} \psi_1(r, z), & 1 < r < \infty, \\ \psi_2(r, z) + \psi^i(r, z), & h < r < 1 \\ \psi_3(r, z), & 0 \leqslant r < h, \end{cases} \tag{2b}$$
$$\psi^i(r, z) = \Psi_{mn}(r)\exp(-\mathrm{i}\omega\mu_{mn}^+ z) \tag{2c}$$
$$\Psi_{mn}(r) = Y_m'(\alpha_{mn}h)J_m(\alpha_{mn}r) - J_m'(\alpha_{mn}h)Y_m(\alpha_{mn}r) \tag{2d}$$

where $\omega > 0$ is the angular frequency and $m \in \mathbb{Z}$ is the circumferential modal order. ψ^i denotes the incident field, which is here assumed to come from the bypass duct, but the analysis would have been similar for the core duct. Radial wave numbers α_{mn}, defined by the eigenvalue equation

$$Y_m'(\alpha_{mn}h)J_m'(\alpha_{mn}) - J_m'(\alpha_{mn}h)Y_m'(\alpha_{mn}) = 0,$$

yield the axial wave numbers $\omega\mu_{mn}^{\pm}$, where

$$\mu_{mn}^{\pm} = \begin{cases} \dfrac{\pm\sqrt{C_1^2 - (1 - M_1^2 C_1^2)\alpha_{mn}^2/\omega^2} - M_1 C_1^2}{1 - M_1^2 C_1^2}, & C_1^2 \geqslant (1 - M_1^2 C_1^2)\alpha_{mn}^2/\omega^2 \quad \text{(cut-on)} \\ \dfrac{\mp \mathrm{i}\sqrt{(1 - M_1^2 C_1^2)\alpha_{mn}^2/\omega^2 - C_1^2} - M_1 C_1^2}{1 - M_1^2 C_1^2}, & \text{otherwise} \quad \text{(cut off)} \end{cases} \tag{3}$$

the signs $(+)$ and $(-)$ denote right and left running modes, respectively. As a special case, we have the plane wave

$$\Psi_{00}(r) = 1 \qquad \text{and} \qquad \mu_{00}^{\pm} = C_1/(M_1 C_1 \pm 1).$$

Time dependence $\sim \mathrm{e}^{\mathrm{i}\omega t}$ and azimuthal dependence $\sim \mathrm{e}^{-\mathrm{i}m\theta}$ are assumed, and suppressed throughout this paper.

2.2 Derivation of the Wiener-Hopf System

The previously introduced velocity potentials $\psi_1(r,z)$, $\psi_2(r,z)$ and $\psi_3(r,z)$ satisfy the Helmholtz equations in respective regions

$$\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial z^2} - \frac{m^2}{r^2} - (\mathrm{i}\omega + M_0\frac{\partial}{\partial z})^2\right]\psi_1(r,z) = 0, \tag{4a}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial z^2} - \frac{m^2}{r^2} - C_1^2(\mathrm{i}\omega + M_1\frac{\partial}{\partial z})^2\right]\psi_2(r,z) = 0, \tag{4b}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial z^2} - \frac{m^2}{r^2} - C_2^2(\mathrm{i}\omega + M_2\frac{\partial}{\partial z})^2\right]\psi_3(r,z) = 0. \tag{4c}$$

Full range Fourier transforms of these equations along z enables us to write solutions as inverse Fourier integrals

$$\psi_1(r,z) = \frac{\omega}{2\pi}\int_L A(u) H_m^{(2)}(\lambda_0\omega r)\,\mathrm{e}^{-\mathrm{i}\omega uz}\,\mathrm{d}u \tag{5a}$$

$$\psi_2(r,z) = \frac{\omega}{2\pi}\int_L [B(u) J_m(\lambda_1\omega r) + C(u) Y_m(\lambda_1\omega r)]\,\mathrm{e}^{-\mathrm{i}\omega uz}\,\mathrm{d}u \tag{5b}$$

$$\psi_3(r,z) = \frac{\omega}{2\pi}\int_L D(u) J_m(\lambda_2\omega r)\,\mathrm{e}^{-\mathrm{i}\omega uz}\,\mathrm{d}u \tag{5c}$$

where L is a suitable integration contour along the real axis in the complex u-domain (see [17]). J_m and Y_m are the Bessel and Neumann functions of order m, $H_m^{(2)} = J_m - \mathrm{i}Y_m$ is the 2nd order Hankel function, compatible with waves outgoing at infinity. The complex square root functions λ_0, λ_1 and λ_2 are defined as follows

$$\lambda_0(u) = \sqrt{(1 - uM_0)^2 - u^2}, \qquad \mathrm{Im}(\lambda_0) \leqslant 0, \tag{6a}$$

$$\lambda_1(u) = \sqrt{C_1^2(1 - uM_1)^2 - u^2}, \quad \mathrm{Im}(\lambda_1) \leqslant 0. \tag{6b}$$

$$\lambda_2(u) = \sqrt{C_2^2(1 - uM_2)^2 - u^2}, \quad \mathrm{Im}(\lambda_2) \leqslant 0. \tag{6c}$$

Branch cuts for λ_0, λ_1 and λ_2 are taken at the real axis along the intervals

$$\left[1/(1+M_0),\infty\right)\cup\left(-\infty,-1/(1-M_0)\right],$$
$$\left[C_2/(1+M_2C_2),\infty\right)\cup\left(-\infty,-C_2/(1-M_2C_2)\right],$$
$$\left[C_2/(1+M_2C_2),\infty\right)\cup\left(-\infty,-C_2/(1-M_2C_2)\right],$$

respectively. The spectral coefficients $A(u)$, $B(u)$, $C(u)$ and $D(u)$, which are introduced in the solutions of velocity potential functions, are to be determined with the aid of the following boundary and continuity relations valid along $r=h$ and $r=1$.

The inner and outer ducts have rigid walls from inside and outside, so that

$$\frac{\partial}{\partial r}\psi_1(1,z)=\frac{\partial}{\partial r}\psi_2(1,z)=0, \qquad z<0, \tag{7a}$$
$$\frac{\partial}{\partial r}\psi_2(h,z)=\frac{\partial}{\partial r}\psi_3(h,z)=0, \qquad z<l, \tag{7b}$$
$$\left(i\omega+M_1\frac{\partial}{\partial z}\right)\eta(z)=\frac{\partial}{\partial r}\psi_2(h,z), \qquad z>l, \tag{7c}$$
$$\left(i\omega+M_2\frac{\partial}{\partial z}\right)\eta(z)=\frac{\partial}{\partial r}\psi_3(h,z), \qquad z>l, \tag{7d}$$
$$\left(i\omega+M_0\frac{\partial}{\partial z}\right)\xi(z)=\frac{\partial}{\partial r}\psi_1(1,z), \qquad z>0, \tag{7e}$$
$$\left(i\omega+M_1\frac{\partial}{\partial z}\right)\xi(z)=\frac{\partial}{\partial r}\psi_2(1,z), \qquad z>0, \tag{7f}$$
$$D_1\left(\mathrm{i}\omega+M_1\frac{\partial}{\partial z}\right)\left[\psi_2(h,z)+\psi^i(h,z)\right]=D_2\left(\mathrm{i}\omega+M_2\frac{\partial}{\partial z}\right)\psi_3(h,z), \quad z>l, \tag{7g}$$
$$D_1\left(\mathrm{i}\omega+M_1\frac{\partial}{\partial z}\right)\left[\psi_2(1,z)+\psi^i(1,z)\right]=\left(\mathrm{i}\omega+M_0\frac{\partial}{\partial z}\right)\psi_1(1,z), \qquad z>0. \tag{7h}$$

The potentials satisfy the usual condition of continuity of particle displacement along $r=1$ and $r=h$. When

$$r=1+\xi(z)\,\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}m\theta} \quad \text{and} \quad r=h+\eta(z)\,\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}m\theta}$$

denotes the complex radial displacement of vortex sheets at $r=1$ and $r=h$, the full Kutta condition applied at either trailing edge implies, respectively,

$$\xi(z)=\mathcal{O}(z^{3/2}) \quad \text{and} \quad \eta(z)=\mathcal{O}(z^{3/2}).$$

Applying the boundary conditions on $r=h$

$$\frac{\partial}{\partial r}\psi_3(h,z)=\frac{\omega}{2\pi}\int_L D(u)\lambda_2\omega J_m'(\lambda_2\omega h)\,\mathrm{e}^{-\mathrm{i}\omega uz}\,\mathrm{d}u \tag{8a}$$
$$\frac{\partial}{\partial r}\psi_2(h,z)=\frac{\omega}{2\pi}\int_L [B(u)\lambda_1\omega J_m'(\lambda_1\omega h)+C(u)\lambda_1\omega Y_m'(\lambda_1\omega h)]\,\mathrm{e}^{-\mathrm{i}\omega uz}\,\mathrm{d}u \tag{8b}$$

and taking Fourier transforms gives

$$D(u)\lambda_2 J_m'(\lambda_2\omega h)=\mathrm{i}(1-uM_2)\,\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^+(u) \tag{9a}$$
$$B(u)\lambda_1 J_m'(\lambda_1\omega h)+C(u)\lambda_1 Y_m'(\lambda_1\omega h)=\mathrm{i}(1-uM_1)\,\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^+(u) \tag{9b}$$

where $\Phi_1^+(u)$ is a function analytic at the upper part of complex $u-$domain and defined as

$$\Phi_1^+(u) = \int_l^\infty \eta(z)\,\mathrm{e}^{\mathrm{i}\omega u(z-l)}\,\mathrm{d}z \tag{10}$$

Continuity of pressure at $r = h$ yields

$$\mathrm{i}\omega\frac{D_2}{D_1}(1-uM_2)D(u)J_m(\lambda_2\omega h) - \mathrm{i}\omega(1-uM_1)\left[B(u)J_m(\lambda_1\omega h) + C(u)Y_m(\lambda_1\omega h)\right] = \\ \mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^-(u) + \mathrm{i}\omega(1-\mu_{mn}^+M_1)\Psi_{mn}(h)\int_l^\infty \mathrm{e}^{\mathrm{i}\omega(u-\mu_{mn}^+)z}\,\mathrm{d}z, \tag{11}$$

which involves unknown coefficients B, C and D. Substituting $D(u)$, given in (9a), into (11) gives us

$$-\mathrm{i}\omega(1-uM_1)\left[B(u)J_m(\lambda_1\omega h) + C(u)Y_m(\lambda_1\omega h)\right] = \mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^-(u) \\ -(1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{\mathrm{i}\omega(u-\mu_{mn}^+)l}}{u-\mu_{mn}^+} + \omega\frac{D_2}{D_1}(1-uM_2)^2\frac{J_m(\lambda_2\omega h)}{\lambda_2 J_m'(\lambda_2\omega h)}\,\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^+(u). \tag{12}$$

A matrix equation can be constructed using the equations (9b) and (12)

$$\begin{bmatrix} \lambda_1 J_m'(\lambda_1\omega h) & \lambda_1 Y_m'(\lambda_1\omega h) \\ -\mathrm{i}\omega(1-uM_1)J_m(\lambda_1\omega h) & -\mathrm{i}\omega(1-uM_1)Y_m(\lambda_1\omega h)\end{bmatrix}\begin{bmatrix} B(u) \\ C(u)\end{bmatrix} = \tag{13}$$

$$\begin{bmatrix} \mathrm{i}(1-uM_1)\,\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^+(u) \\ \mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^-(u) - (1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{\mathrm{i}\omega(u-\mu_{mn}^+)l}}{u-\mu_{mn}^+} + \omega\frac{D_2}{D_1}(1-uM_2)^2\frac{J_m(\lambda_2\omega h)}{\lambda_2 J_m'(\lambda_2\omega ht)}\,\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^+(u)\end{bmatrix},$$

which allows us to express the unknown coefficients $B(u)$ and $C(u)$ in terms of the analytic functions $\Phi_1^+(u)$ and $\Phi_1^-(u)$ as follows.

$$B(u) = \frac{\pi h}{2\mathrm{i}(1-uM_1)}\times \\ \left\{\omega\,\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^+(u)\left[(1-uM_1)^2Y_m(\lambda_1\omega h) - \frac{D_2}{D_1}(1-uM_2)^2\frac{J_m(\lambda_2\omega h)\lambda_1 Y_m'(\lambda_1\omega h)}{\lambda_2 J_m'(\lambda_2\omega h)}\right]\right. \\ \left. -\left[\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^-(u) - (1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{\mathrm{i}\omega(u-\mu_{mn}^+)l}}{u-\mu_{mn}^+}\right]\lambda_1 Y_m'(\lambda_1\omega h)\right\} \tag{14a}$$

$$C(u) = \frac{-\pi h}{2\mathrm{i}(1-uM_1)}\times \\ \left\{\omega\,\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^+(u)\left[(1-uM_1)^2J_m(\lambda_1\omega h) - \frac{D_2}{D_1}(1-uM_2)^2\frac{J_m(\lambda_2\omega h)\lambda_1 J_m'(\lambda_1\omega h)}{\lambda_2 J_m'(\lambda_2\omega h)}\right]\right. \\ \left. -\left[\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^-(u) - (1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{\mathrm{i}\omega(u-\mu_{mn}^+)l}}{u-\mu_{mn}^+}\right]\lambda_1 J_m'(\lambda_1\omega h)\right\} \tag{14b}$$

where $\Phi_1^-(u)$ is a function analytic on the lower part of the complex $u-$domain and defined as

$$\Phi_1^-(u) = \int_{-\infty}^{l} \left[\frac{D_2}{D_1}(\mathrm{i}\omega + M_2\frac{\partial}{\partial z})\psi_3(h,z) - (\mathrm{i}\omega + M_1\frac{\partial}{\partial z})\psi_2(h,z) \right] \mathrm{e}^{\mathrm{i}\omega u(z-l)}\, \mathrm{d}z. \tag{15}$$

Similarly, application of the boundary conditions on $r = 1$

$$\frac{\partial}{\partial r}\psi_1(1,z) = \frac{\omega}{2\pi}\int_L A(u)\lambda_0\omega H_m^{(2)\prime}(\lambda_0\omega)\,\mathrm{e}^{-\mathrm{i}\omega uz}\, \mathrm{d}u, \tag{16a}$$

$$\frac{\partial}{\partial r}\psi_2(1,z) = \frac{\omega}{2\pi}\int_L [B(u)\lambda_1\omega J_m'(\lambda_1\omega) + C(u)\lambda_1\omega Y_m'(\lambda_1\omega)]\,\mathrm{e}^{-\mathrm{i}\omega uz}\, \mathrm{d}u, \tag{16b}$$

results into

$$A(u)\lambda_0 H_m^{(2)\prime}(\lambda_0\omega) = \mathrm{i}(1-uM_0)\Phi_2^+(u), \tag{17a}$$

$$B(u)\lambda_1 J_m'(\lambda_1\omega) + C(u)\lambda_1 Y_m'(\lambda_1\omega) = \mathrm{i}(1-uM_1)\Phi_2^+(u). \tag{17b}$$

Substituting $B(u)$ and $C(u)$ in (14a,14b) we get

$$\begin{aligned}
&\frac{\pi\omega h\,\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^+(u)}{2\mathrm{i}(1-uM_1)}\Bigg\{(1-uM_1)^2\lambda_1[Y_m(\lambda_1\omega h)J_m'(\lambda_1\omega) - J_m(\lambda_1\omega h)Y_m'(\lambda_1\omega)] \\
&\quad -\frac{D_2}{D_1}(1-uM_2)^2\frac{\lambda_1^2 J_m(\lambda_2\omega h)}{\lambda_2 J_m'(\lambda_2\omega h)}[Y_m'(\lambda_1\omega h)J_m'(\lambda_1\omega) - J_m'(\lambda_1\omega h)Y_m'(\lambda_1\omega)]\Bigg\} \\
&\quad - \frac{\lambda_1^2\pi h}{2\mathrm{i}(1-uM_1)}\left[\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^-(u) - (1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{\mathrm{i}\omega(u-\mu_{mn}^+)l}}{u-\mu_{mn}^+}\right]\times \\
&\quad \Big[Y_m'(\lambda_1\omega h)J_m'(\lambda_1\omega) - J_m'(\lambda_1\omega h)Y_m'(\lambda_1\omega)\Big] = \mathrm{i}(1-uM_1)\Phi_2^+(u).
\end{aligned} \tag{18}$$

Continuity of pressure at $r = 1$ together with (14a,14b) and (17a) yields

$$\begin{aligned}
&-\omega(1-uM_0)^2\frac{H_m^{(2)}(\lambda_0\omega)}{\lambda_0 H_m^{(2)\prime}(\lambda_0\omega)}\Phi_2^+(u) \\
&\quad - D_1\frac{\pi\omega^2 h}{2}\,\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^+(u)\Bigg\{(1-uM_1)^2[Y_m(\lambda_1\omega h)J_m(\lambda_1\omega) - J_m(\lambda_1\omega h)Y_m(\lambda_1\omega)] \\
&- \frac{D_2}{D_1}(1-uM_2)^2\frac{J_m(\lambda_2\omega h)}{\lambda_2 J_m'(\lambda_2\omega h)}\lambda_1[Y_m'(\lambda_1\omega h)J_m(\lambda_1\omega) - J_m'(\lambda_1\omega h)Y_m(\lambda_1\omega)]\Bigg\} + D_1\frac{\pi\omega h}{2}\times \\
&\left[\mathrm{e}^{\mathrm{i}\omega ul}\,\Phi_1^-(u) - (1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{\mathrm{i}\omega(u-\mu_{mn}^+)l}}{u-\mu_{mn}^+}\right]\lambda_1[Y_m'(\lambda_1\omega h)J_m(\lambda_1\omega) - J_m'(\lambda_1\omega h)Y_m(\lambda_1\omega)] \\
&\quad = \Phi_2^-(u) + \mathrm{i}\omega\Psi_{mn}(1)D_1(1-\mu_{mn}^+M_1)\int_0^\infty \exp[\mathrm{i}\omega(u-\mu_{mn}^+)z]\,\mathrm{d}z.
\end{aligned} \tag{19}$$

In the above equations, $\Phi_2^+(u)$ and $\Phi_2^-(u)$ are defined by

$$\Phi_2^+(u) = \int_0^\infty \xi(z)\,\mathrm{e}^{\mathrm{i}\omega uz}\, \mathrm{d}z, \tag{20a}$$

$$\Phi_2^-(u) = \int_{-\infty}^0 \left[\Big(\mathrm{i}\omega + M_0\frac{\partial}{\partial z}\Big)\psi_1(1,z) - D_1\Big(\mathrm{i}\omega + M_1\frac{\partial}{\partial z}\Big)\psi_2(1,z)\right]\mathrm{e}^{\mathrm{i}\omega uz}\, \mathrm{d}z. \tag{20b}$$

By rearranging the first equation and eliminating $\Phi_1^+(u)$ in the second, while using the first one, we obtain a couple of Wiener-Hopf equations to be solved

$$\Phi_1^+(u) - \frac{\lambda_1\lambda_2}{\omega N(u)}\left[\Phi_1^-(u) - (1-\mu_{mn}^+ M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+ l}}{u-\mu_{mn}^+}\right] = -\frac{2}{\pi\omega h}(1-uM_1)^2\frac{\mathrm{e}^{-\mathrm{i}\omega ul}\,\Phi_2^+(u)}{\chi_1(u)}, \tag{21a}$$

$$\omega\Phi_2^+(u)K(u) - \frac{2}{\pi\omega}\frac{D_1(1-uM_1)^2}{\chi_1(u)}\,\mathrm{e}^{\mathrm{i}\omega ul}\left[\Phi_1^-(u) - (1-\mu_{mn}^+ M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+ l}}{u-\mu_{mn}^+}\right] = \Phi_2^-(u) - \Psi_{mn}(1)D_1\frac{(1-\mu_{mn}^+ M_1)}{u-\mu_{mn}^+}, \tag{21b}$$

where $\chi_1(u)$ and $\chi_2(u)$ are analytic functions except for some simple zeros and poles

$$\chi_1(u) = (1-uM_1)^2\lambda_1[Y_m(\lambda_1\omega h)J_m'(\lambda_1\omega) - J_m(\lambda_1\omega h)Y_m'(\lambda_1\omega)] - \frac{D_2}{D_1}(1-uM_2)^2\frac{J_m(\lambda_2\omega h)}{\lambda_2 J_m'(\lambda_2\omega h)}\lambda_1^2[Y_m'(\lambda_1\omega h)J_m'(\lambda_1\omega) - J_m'(\lambda_1\omega h)Y_m'(\lambda_1\omega)], \tag{22a}$$

$$\chi_2(u) = (1-uM_1)^2[Y_m(\lambda_1\omega h)J_m(\lambda_1\omega) - J_m(\lambda_1\omega h)Y_m(\lambda_1\omega)] - \frac{D_2}{D_1}(1-uM_2)^2\frac{J_m(\lambda_2\omega h)}{\lambda_2 J_m'(\lambda_2\omega h)}\lambda_1[Y_m'(\lambda_1\omega h)J_m(\lambda_1\omega) - J_m'(\lambda_1\omega h)Y_m(\lambda_1\omega)]. \tag{22b}$$

$N(u)$ and $K(u)$ are kernel functions which have to be factorised. They are defined by

$$N(u) = \lambda_2(1-uM_1)^2\frac{Y_m(\lambda_1\omega h)J_m'(\lambda_1\omega) - J_m(\lambda_1\omega h)Y_m'(\lambda_1\omega)}{Y_m'(\lambda_1\omega h)J_m'(\lambda_1\omega) - J_m'(\lambda_1\omega h)Y_m'(\lambda_1\omega)} - \lambda_1\frac{D_2}{D_1}(1-uM_2)^2\frac{J_m(\lambda_2\omega h)}{J_m'(\lambda_2\omega h)}, \tag{23a}$$

$$K(u) = D_1(1-uM_1)^2\frac{\chi_2(u)}{\chi_1(u)} - (1-uM_0)^2\frac{H_m^{(2)}(\lambda_0\omega)}{\lambda_0 H_m^{(2)\prime}(\lambda_0\omega)}. \tag{23b}$$

2.3 Kernel functions $K(u)$ and $N(u)$

Unlike the usual Wiener-Hopf problems which normally includes one discontinuity along the boundary, we have now two kernel functions.

The first kernel $N(u)$ is a result of the interaction between the jet and the core flow. The points where the function is zero or infinite provide us information on both mathematical and physical sides of the problem. The poles can be found numerically where the denominator

$$J_m'(\lambda_2\omega h)\big(Y_m'(\lambda_1\omega h)J_m'(\lambda_1\omega) - J_m'(\lambda_1\omega h)Y_m'(\lambda_1\omega)\big)$$

is zero. The first term is the characteristic equation of the infinite annular duct, and the second term is the characteristic equation of the infinite hollow duct. Thus the poles of $N(u)$ correspond to the acoustic modes of the ducts. Poles and a group of zeros are situated closely to the lines

$$u_c = -\frac{M_1C_1^2}{1-M_1^2C_1^2} \quad \text{and} \quad u_d = -\frac{M_2C_2^2}{1-M_2^2C_2^2},$$

which are on the negative side of the real axis. One group of zeros is more related with the behaviour of the vortex sheet. One of these zeros is the Kelvin-Helmholtz instability mode along the inner vortex sheet. A high frequency approximation to get an analytic expressions for this zero of $N(u)$ can be used. It is found as

$$u_1 = \left\{ \frac{M_1 + M_2}{2} + \frac{1}{2}\left[(M_1 - M_2)^2 + 4 - 4\sqrt{1 + (M_1 - M_2)^2} \right]^{1/2} \right\}^{-1}. \tag{24a}$$

Similar to $N(u)$, the kernel $K(u)$ also has a group of zeros and poles closely located to the lines

$$u_c = -\frac{M_1 C_1^2}{1 - M_1^2 C_1^2} \quad \text{and} \quad u_d = -\frac{M_2 C_2^2}{1 - M_2^2 C_2^2}.$$

One group of poles of $K(u)$, which are the solution of the equation $\chi_1(u) = 0$, is related with the annular duct and infinite duct modes, while the one group of zeros is connected with the vortex sheet of the outer duct. The zero, which corresponds to the instability of the outer vortex sheet, can be approximated by the formula

$$u_0 = \left\{ \frac{M_0 + M_1}{2} + \frac{1}{2}\left[(M_0 - M_1)^2 + 4 - 4\sqrt{1 + (M_0 - M_1)^2} \right]^{1/2} \right\}^{-1}. \tag{24b}$$

The kernel functions have the following asymptotic behaviour when $|u| \to \infty$,

$$N(u) \sim \mathcal{O}(u^3), \qquad K(u) \sim \mathcal{O}(u). \tag{25}$$

If the flow speeds are zero, i.e. $M_0 = M_1 = M_2 = 0$, the corresponding behaviour becomes

$$N(u) \sim \mathcal{O}(u), \qquad K(u) \sim \mathcal{O}(u^{-1}). \tag{26}$$

2.4 Wiener Hopf Solution with Weak Factorization

Consider the first equation in (21a) and rearrange it in the following form

$$\begin{aligned}
\frac{\Phi_1^+(u) N_+(u)}{\lambda_1^+(u)\lambda_2^+(u)} &+ \frac{2}{\pi \omega h}(1 - uM_1)^2 \frac{\mathrm{e}^{-\mathrm{i}\omega u l}\, \Phi_2^+(u) N_+(u)}{\chi_1(u)\lambda_1^+(u)\lambda_2^+(u)} \\
&+ (1 - \mu_{mn}^+ M_1)\Psi_{mn}(h) \frac{N_-(\mu_{mn}^+)\lambda_1^-(\mu_{mn}^+)\lambda_2^-(\mu_{mn}^+)\,\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+ l}}{\omega(u - \mu_{mn}^+)} = \\
&\frac{N_-(u)\lambda_1^-(u)\lambda_2^-(u)}{\omega}\left[\Phi_1^-(u) - (1 - \mu_{mn}^+ M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+ l}}{u - \mu_{mn}^+} \right] \\
&\qquad + (1 - \mu_{mn}^+ M_1)\Psi_{mn}(h) \frac{N_-(\mu_{mn}^+)\lambda_1^-(\mu_{mn}^+)\lambda_2^-(\mu_{mn}^+)\,\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+ l}}{\omega(u - \mu_{mn}^+)}.
\end{aligned} \tag{27}$$

Here, $N_+(u)$ and $N_-(u)$ are the split functions regular and free of zeros in the upper and lower half planes, respectively, resulting from the Wiener-Hopf factorization of $N(u)$

$$N(u) = \frac{N_+(u)}{N_-(u)} \tag{28}$$

Now, the right hand side is a function analytic in the lower half of the complex $u-$domain and the left hand side is a function analytic in the upper half of the complex $u-$domain, except for the zeros of the function $\chi_1(u)$. These zeros β_{mn}^-, which belong to the upper half plane, satisfy the equation

$$\chi_1(\beta_{mn}^-) = 0, \qquad n = 1, 2, \dots \tag{29}$$

If the infinite sum of poles is subtracted from both sides of the equation we have

$$\begin{aligned}
&\frac{\Phi_1^+(u)N_+(u)}{\lambda_1^+(u)\lambda_2^+(u)} + \frac{2}{\pi\omega h}(1-uM_1)^2\frac{\mathrm{e}^{-\mathrm{i}\omega ul}\,\Phi_2^+(u)N_+(u)}{\chi_1(u)\lambda_1^+(u)\lambda_2^+(u)}\\
&\quad + (1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{N_-(\mu_{mn}^+)\lambda_1^-(\mu_{mn}^+)\lambda_2^-(\mu_{mn}^+)\,\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{\omega(u-\mu_{mn}^+)} - \sum_{p=1}^{\infty}\frac{a_{mp}}{u-\beta_{mp}^-} =\\
&\quad \frac{N_-(u)\lambda_1^-(u)\lambda_2^-(u)}{\omega}\left[\Phi_1^-(u) - (1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{u-\mu_{mn}^+}\right]\\
&\quad + (1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{N_-(\mu_{mn}^+)\lambda_1^-(\mu_{mn}^+)\lambda_2^-(\mu_{mn}^+)\,\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{\omega(u-\mu_{mn}^+)} - \sum_{p=1}^{\infty}\frac{a_{mp}}{u-\beta_{mp}^-},
\end{aligned} \tag{30a}$$

with

$$a_{mp} = \frac{2}{\pi\omega h}(1-\beta_{mp}^-M_1)^2\frac{\mathrm{e}^{-\mathrm{i}\omega\beta_{mp}^-l}\,\Phi_2^+(\beta_{mp}^-)N_+(\beta_{mp}^-)}{\chi_1'(\beta_{mp}^-)\lambda_1^+(\beta_{mp}^-)\lambda_2^+(\beta_{mp}^-)}. \tag{30b}$$

Application of the analytic continuation principle and Liouville's theorem, together with a Kutta condition (controlled by parameter Γ_1) at $r = h$, $z \downarrow l$ yields

$$\begin{aligned}
&\frac{N_-(u)\lambda_1^-(u)\lambda_2^-(u)}{\omega}\left[\Phi_1^-(u) - (1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{u-\mu_{mn}^+}\right] =\\
&\quad -(1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{N_-(\mu_{mn}^+)\lambda_1^-(\mu_{mn}^+)\lambda_2^-(\mu_{mn}^+)\,\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{\omega(\mu_{mn}^+-u_1)}\left[\frac{u-u_1}{u-\mu_{mn}^+}-\Gamma_1\right]\\
&\quad + \sum_{p=1}^{\infty}\frac{a_{mp}}{\beta_{mp}^--u_1}\left[\frac{u-u_1}{u-\beta_{mp}^-}-\Gamma_1\right],
\end{aligned} \tag{31a}$$

$$\begin{aligned}
&\frac{\Phi_1^+(u)N_+(u)}{\lambda_1^+(u)\lambda_2^+(u)} = -\frac{2}{\pi\omega h}(1-uM_1)^2\frac{\mathrm{e}^{-\mathrm{i}\omega ul}\,\Phi_2^+(u)N_+(u)}{\chi_1(u)\lambda_1^+(u)\lambda_2^+(u)}\\
&\quad -(1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{N_-(\mu_{mn}^+)\lambda_1^-(\mu_{mn}^+)\lambda_2^-(\mu_{mn}^+)\,\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{\omega(\mu_{mn}^+-u_1)}\left[\frac{u-u_1}{u-\mu_{mn}^+}-\Gamma_1\right]\\
&\quad + \sum_{p=1}^{\infty}\frac{a_{mp}}{\beta_{mp}^--u_1}\left[\frac{u-u_1}{u-\beta_{mp}^-}-\Gamma_1\right].
\end{aligned} \tag{31b}$$

$\Gamma_1 = 1$ corresponds to the full Kutta condition at the inner trailing edge, and $\Gamma_1 = 0$ to no Kutta condition such that no vorticity is shed.

Consider now the second W-H equation given in (21b) and perform the usual Wiener-Hopf factorisation and decomposition procedures, so that we arrive at the equation

$$\begin{aligned}&\omega\Phi_2^+(u)K_+(u)+\delta(m,n)\frac{D_1(1-\mu_{mn}^+M_1)}{u-\mu_{mn}^+}K_-(\mu_{mn}^+)\left\{\frac{2}{\pi\omega}\frac{(1-\mu_{mn}^+M_1)^2}{\chi_1(\mu_{mn}^+)}+1\right\}=\\&\quad\frac{2}{\pi\omega}\frac{D_1(1-uM_1)^2K_-(u)}{\chi_1(u)}\,\mathrm{e}^{\mathrm{i}\omega ul}\left[\Phi_1^-(u)-(1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{u-\mu_{mn}^+}\right]+\Phi_2^-(u)K_-(u)\\&-D_1\Psi_{mn}(1)\frac{(1-\mu_{mn}^+M_1)K_-(u)}{u-\mu_{mn}^+}+\delta(m,n)\frac{D_1(1-\mu_{mn}^+M_1)}{u-\mu_{mn}^+}K_-(\mu_{mn}^+)\left\{\frac{2}{\pi\omega}\frac{(1-\mu_{mn}^+M_1)^2}{\chi_1(\mu_{mn}^+)}+1\right\},\end{aligned} \tag{32}$$

where $\delta(m,n)$ is defined[1] as

$$\delta(m,n)=\begin{cases}1 & , \quad m=0,n=0\\ 0 & , \quad \text{otherwise.}\end{cases} \tag{33}$$

The split functions $K_+(u)$ and $K_-(u)$, result from the factorization of $K(u)$, regular and free of zeros in the upper and lower half planes, respectively, as follows

$$K(u)=\frac{K_+(u)}{K_-(u)}. \tag{34}$$

The regularity of the right hand side is violated by the zeros on the lower half plane, namely at β_{mn}^+, with

$$\chi_1(\beta_{mn}^+)=0,\ \beta_{m0}^+=u_1. \tag{35}$$

Subtraction of the infinite sum of poles from both sides, yields

$$\begin{aligned}&\omega\Phi_2^+(u)K_+(u)-\sum_{p=1}^{\infty}\frac{b_{mp}}{u-\beta_{mp}^+}-\frac{b_{m0}}{u-u_1}\\&\qquad+\delta(m,n)\frac{D_1(1-\mu_{mn}^+M_1)}{u-\mu_{mn}^+}K_-(\mu_{mn}^+)\left\{\frac{2}{\pi\omega}\frac{(1-\mu_{mn}^+M_1)^2}{\chi_1(\mu_{mn}^+)}+1\right\}=\\&\quad\delta(m,n)\frac{D_1(1-\mu_{mn}^+M_1)}{u-\mu_{mn}^+}K_-(\mu_{mn}^+)\left\{\frac{2}{\pi\omega}\frac{(1-\mu_{mn}^+M_1)^2}{\chi_1(\mu_{mn}^+)}+1\right\}+\Phi_2^-(u)K_-(u)\\&\quad+\frac{2}{\pi\omega}\frac{D_1(1-uM_1)^2K_-(u)}{\chi_1(u)}\,\mathrm{e}^{\mathrm{i}\omega ul}\left\{\Phi_1^-(u)-(1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{u-\mu_{mn}^+}\right\}\\&\qquad-D_1\Psi_{mn}(1)\frac{(1-\mu_{mn}^+M_1)K_-(u)}{u-\mu_{mn}^+}-\sum_{p=1}^{\infty}\frac{b_{mp}}{u-\beta_{mp}^+}-\frac{b_{m0}}{u-u_1},\end{aligned} \tag{36a}$$

where

$$b_{m0}=\frac{2}{\pi\omega}\frac{D_1(1-u_1M_1)^2K_-(u_1)}{\chi_1'(u_1)}\,\mathrm{e}^{\mathrm{i}\omega u_1l}\left\{\Phi_1^-(u_1)-(1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{u_1-\mu_{mn}^+}\right\}, \tag{36b}$$

$$b_{mp}=\frac{2}{\pi\omega}\frac{D_1(1-\beta_{mp}^+M_1)^2K_-(\beta_{mp}^+)}{\chi_1'(\beta_{mp}^+)}\,\mathrm{e}^{\mathrm{i}\omega\beta_{mp}^+l}\left\{\Phi_1^-(\beta_{mp}^+)-(1-\mu_{mn}^+M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+l}}{\beta_{mp}^+-\mu_{mn}^+}\right\}. \tag{36c}$$

[1] If $\delta_{m,n}$ denotes the Kronecker delta, we could also write $\delta(m,n)=\delta_{|m|+|n|,0}$.

b_{m0} is zero when $M_1 = M_2$, which can be seen from the formula (36b).

The solution to the second Wiener-Hopf equation, using the analytic continuation principle and Liouville's theorem, is then eventually:

$$\Phi_2^-(u) = -\frac{2}{\pi\omega}\frac{D_1(1-uM_1)^2}{\chi_1(u)}\,\mathrm{e}^{\mathrm{i}\omega ul}\left\{\Phi_1^-(u) - (1-\mu_{mn}^+ M_1)\Psi_{mn}(h)\frac{\mathrm{e}^{-\mathrm{i}\omega\mu_{mn}^+ l}}{u-\mu_{mn}^+}\right\}$$
$$+ D_1\Psi_{mn}(1)\frac{1-\mu_{mn}^+ M_1}{u-\mu_{mn}^+} + \frac{1}{K_-(u)}\sum_{p=1}^{\infty}\frac{b_{mp}}{\beta_{mp}^+ - u_0}\left[\frac{u-u_0}{u-\beta_{mp}^+} - \Gamma_0\right] + \frac{1}{K_-(u)}\frac{b_{m0}}{u_1-u_0}\left[\frac{u-u_0}{u-u_1} - \Gamma_0\right]$$
$$- \delta(m,n)\frac{D_1(1-\mu_{00}^+ M_1)}{(\mu_{00}^+ - u_0)K_-(u)}K_-(\mu_{00}^+)\left\{\frac{2}{\pi\omega}\frac{(1-\mu_{00}^+ M_1)^2}{\chi_1(\mu_{00}^+)} + 1\right\}\left[\frac{u-u_0}{u-\mu_{00}^+} - \Gamma_0\right], \quad (37a)$$

and

$$\omega\Phi_2^+(u)K_+(u) = \sum_{p=1}^{\infty}\frac{b_{mp}}{\beta_{mp}^+ - u_0}\left[\frac{u-u_0}{u-\beta_{mp}^+} - \Gamma_0\right] + \frac{b_{m0}}{u_1-u_0}\left[\frac{u-u_0}{u-\beta_{mp}^+} - \Gamma_0\right]$$
$$- \delta(m,n)\frac{D_1(1-\mu_{00}^+ M_1)}{\mu_{00}^+ - u_0}K_-(\mu_{00}^+)\left\{\frac{2}{\pi\omega}\frac{(1-\mu_{00}^+ M_1)^2}{\chi_1(\mu_{00}^+)} + 1\right\}\left[\frac{u-u_0}{u-\mu_{00}^+} - \Gamma_0\right], \quad (37b)$$

where $\Gamma_0 = 1$ corresponds to the full Kutta condition, while $\Gamma_0 = 0$ corresponds to no Kutta condition, such that no vorticity is shed, at the outer trailing edge.

We can determine the unknown coefficients a_{mp} and b_{mp}, which are seen in the solution of the W-H system, by solving the following linear algebraic system

$$\frac{\pi}{2}\frac{\mathrm{e}^{-i\omega u_1 l}\,\chi_1'(u_1)N_-(u_1)\lambda_1^-(u_1)\lambda_2^-(u_1)}{D_1(1-u_1M_1)^2K_-(u_1)}b_{m0} = -\Gamma_1\sum_{p=1}^{\infty}\frac{a_{mp}}{\beta_{mp}^- - u_1}$$
$$+ \Gamma_1(1-\mu_{mn}^+ M_1)\Psi_{mn}(h)\frac{N_-(\mu_{mn}^+)\lambda_1^-(\mu_{mn}^+)\lambda_2^-(\mu_{mn}^+)\,\mathrm{e}^{-i\omega\mu_{mn}^+ l}}{\omega(\mu_{mn}^+ - u_1)} \quad (38a)$$

$$\frac{\pi}{2}\frac{\mathrm{e}^{-\mathrm{i}\omega\beta_{mr}^+ l}\,\chi_1'(\beta_{mr}^+)N_-(\beta_{mr}^+)\lambda_1^-(\beta_{mr}^+)\lambda_2^-(\beta_{mr}^+)}{D_1(1-\beta_{mr}^+ M_1)^2K_-(\beta_{mr}^+)}b_{mr} = \sum_{p=1}^{\infty}\frac{a_{mp}}{\beta_{mp}^- - u_1}\left[\frac{\beta_{mr}^+ - u_1}{\beta_{mr}^+ - \beta_{mp}^-} - \Gamma_1\right]$$
$$- (1-\mu_{mn}^+ M_1)\Psi_{mn}(h)\frac{N_-(\mu_{mn}^+)\lambda_1^-(\mu_{mn}^+)\lambda_2^-(\mu_{mn}^+)e^{-i\omega\mu_{mn}^+ l}}{\omega(\mu_{mn}^+ - u_1)}\left[\frac{\beta_{mr}^+ - u_1}{\beta_{mr}^+ - \mu_{mn}^+} - \Gamma_1\right] \quad (38b)$$

$$\frac{\pi\omega^2 h}{2}\frac{\mathrm{e}^{\mathrm{i}\omega\beta_{mr}^- l}\,\chi_1'(\beta_{mr}^-)K_+(\beta_{mr}^-)\lambda_1^+(\beta_{mr}^-)\lambda_2^+(\beta_{mr}^-)}{(1-\beta_{mr}^- M_1)^2N_+(\beta_{mr}^-)}a_{mr} =$$
$$\sum_{p=1}^{\infty}\frac{b_{mp}}{\beta_{mp}^+ - u_0}\left[\frac{\beta_{mr}^- - u_0}{\beta_{mr}^- - \beta_{mp}^+} - \Gamma_0\right] + \frac{b_{m0}}{u_1-u_0}\left[\frac{\beta_{mr}^- - u_0}{\beta_{mr}^- - u_1} - \Gamma_0\right]$$
$$- \delta(m,n)\frac{D_1(1-\mu_{00}^+ M_1)}{\mu_{00}^+ - u_0}K_-(\mu_{00}^+)\left\{\frac{2}{\pi\omega}\frac{(1-\mu_{00}^+ M_1)^2}{\chi_1(\mu_{00}^+)} + 1\right\}\left[\frac{\beta_{mr}^- - u_0}{\beta_{mr}^- - \mu_{00}^+} - \Gamma_0\right]. \quad (38c)$$

Note that in (38a) the coefficient b_{m0} vanishes when $\Gamma_1 = 0$, i.e. when no Kutta condition is applied and no vorticity is shed from the inner edge. Because of the multiplication with $\mathrm{e}^{-\mathrm{i}\omega u_1 l}$, b_{m0} becomes exponentially large for $l < 0$. This will be seen to have a dramatic effect in the far field [21].

2.5 Factorization of the Kernels $K(u)$, $N(u)$

The kernel functions have possible zeros which causally belong to the lower half plane, but are located on the upper half plane. We can bring these zeros to the other side by adding a factor $(u-u_i)$, $i=0,1$ in the definition of split functions. Hence, we can write the split functions now as

$$N_{\pm}(u) = \tilde{N}_{\pm}(u)(u-u_1), \tag{39a}$$

$$K_{\pm}(u) = \tilde{K}_{\pm}(u)(u-u_0), \tag{39b}$$

where $\tilde{N}_{\pm}(u)$ and $\tilde{K}_{\pm}(u)$ are to be calculated by the following integrals

$$\log \tilde{N}_{\pm}(\xi) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\log[N(u)]}{u-\xi}\, \mathrm{d}u, \tag{40a}$$

$$\log \tilde{K}_{\pm}(\xi) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\log[K(u)]}{u-\xi}\, \mathrm{d}u. \tag{40b}$$

The $+$ sign corresponds with $\operatorname{Im}\xi > \operatorname{Im}\alpha$ for all $\alpha \in L$, and the $-$ sign with $\operatorname{Im}\xi < \operatorname{Im}\alpha$ for all $\alpha \in L$.

The split functions have the following asymptotic behaviour

$$\tilde{N}_{\pm}(u) \sim \mathcal{O}(u^{\pm 3/2}), \tag{41a}$$

$$\tilde{K}_{\pm}(u) \sim \mathcal{O}(u^{\pm 1/2}). \tag{41b}$$

If the instability zeros do not cross the contour L, we define the split functions

$$N_{\pm}(u) = \tilde{N}_{\pm}(u), \tag{42a}$$

$$K_{\pm}(u) = \tilde{K}_{\pm}(u). \tag{42b}$$

3 Far Field

The pressure in the far field can be calculated in terms of the following integral

$$p_1(r,z) = \frac{\omega^2}{2\pi} \int_L \frac{(1-uM_0)^2 \Phi_2^+(u)}{\lambda_0} \frac{H_m^{(2)}(\lambda_0 \omega r)}{H_m^{(2)\prime}(\lambda_0 \omega)}\, \mathrm{e}^{-\mathrm{i}\omega u z}\, \mathrm{d}u, \tag{43}$$

where L is the inverse Fourier transform contour.

Using the asymptotic formula for Hankel functions while $\omega r \gg 1$ in the integral given with (43), we get

$$p_1(r,z) \sim \frac{\omega^2}{2\pi} \int_L \frac{(1-uM_0)^2 \Phi_2^+(u)}{\lambda_0 H_m^{(2)\prime}(\lambda_0 \omega)} \sqrt{\frac{2}{\pi \lambda_0 \omega r}}\, \mathrm{e}^{-\mathrm{i}(\omega \lambda_0 r + \omega u z - \frac{1}{2} m\pi - \frac{1}{4}\pi)}\, \mathrm{d}u. \tag{44}$$

We convert the integral in a form to which the method of steepest descent is applicable, by the transformations

$$r = R\sin\theta (1-M_0^2)^{1/2}, \quad z = R\cos\theta(1-M_0^2), \quad \text{and} \quad u = \frac{\cos(\theta - i\tau) - M_0}{\left(1-M_0^2\right)}.$$

The latter transformation deforms the integration contour into a branch of a hyperbola. Now the solution can be approximated by

$$p_1(R,\theta) \sim \frac{\mathrm{i}\omega^2}{\pi} \frac{(1-uM_0)^2 \Phi_2^+(u)}{(1-M_0^2)^{1/2} \sin\theta\, H_m^{(2)\prime}(\lambda_0 \omega)}\, \mathrm{e}^{\frac{1}{2} m\pi \mathrm{i}}\, \frac{\mathrm{e}^{-\mathrm{i}\omega R(1-M_0\cos\theta)}}{\omega R}. \tag{45}$$

After converting to the original variables,

$$z = \bar{R}\cos\bar{\theta}, \qquad r = \bar{R}\sin\bar{\theta}, \tag{46}$$

we obtain

$$p_1(\bar{R},\bar{\theta}) \sim \frac{D_p(\bar{\theta})}{\bar{R}}\, \mathrm{e}^{-\mathrm{i}\omega\overline{R}[-M_0\cos\bar{\theta}+(1-M_0^2\sin^2\bar{\theta})^{1/2}]/(1-M_0^2)+\mathrm{i}\frac{1}{2}(m+1)\pi} \tag{47}$$

where directivity D_p of the pressure field is given by

$$D_p(\bar{\theta}) = \frac{\omega}{\pi}\frac{(1-u'M_0)^2\Phi_2^+(u')}{\sin\bar{\theta}\,H_m^{(2)\prime}(\lambda_0'\omega)}. \tag{48}$$

and u' and λ_0' are defined by

$$u' = \frac{\cos\bar{\theta}(1-M_0^2\sin^2\bar{\theta})^{-1/2}-M_0}{1-M_0^2}, \quad \lambda_0' = \frac{\sin\bar{\theta}}{(1-M_0^2\sin^2\bar{\theta})^{1/2}}. \tag{49}$$

Note that we did not include a near field effect due to a wave captured by the jet, which, as reported by Gabard et al. [23], extends in many practical cases with mean flow into the far field near the jet.

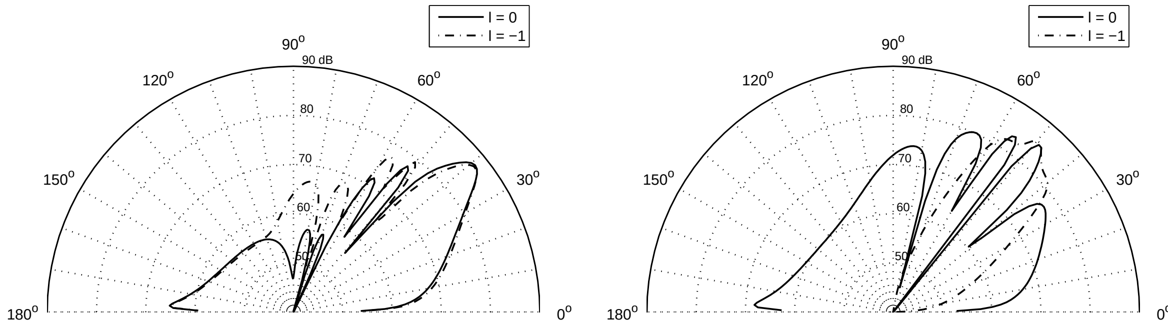
4 Numerical Examples

A series of examples are numerically evaluated to investigate the effects of buried versus co-planar nozzle, with and without mean flow and with and without Kutta condition. These examples are mainly meant to illustrate, because it is not possible here to make an exhaustive investigation of all possible combinations of problem parameters nor scan all the physical consequences. The problem parameters that were used are:

Table 1: Problem parameters for far field examples.

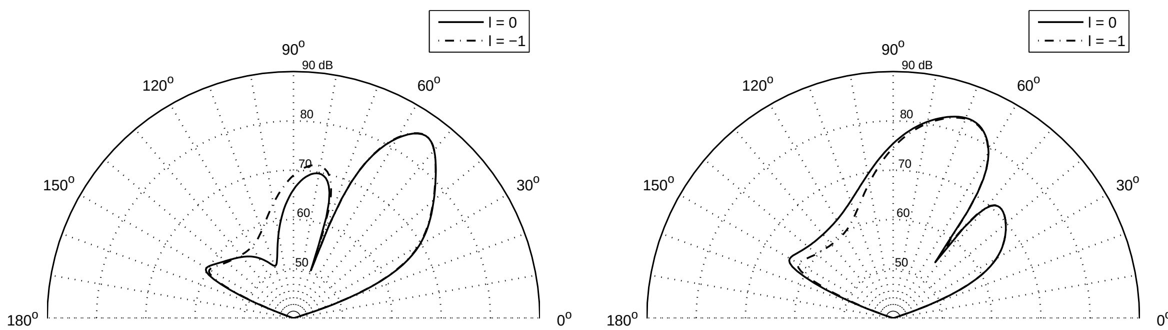
parameter	definition	to calculate	value
ω	$2\pi f R_2/c_0$	$2\pi\cdot 876\cdot 1.2/330$	20.0
m			1 and 10
M_0	U_0/c_0	99/330	0.30
M_1	U_j/c_0	210/330	0.64
M_2	U_c/c_0	265/330	0.80
C_1	c_0/c_j	330/350	0.94
C_2	c_0/c_c	330/530	0.62
D_1	ρ_j/ρ_0	1.158/1.3026	0.89
D_2	ρ_c/ρ_j	0.505/1.158	0.44
h	R_1/R_2	0.8/1.2	0.67
l	inner duct exit plane	$-5R_2 \leqslant lR_2 \leqslant 0$	0, −1, −2, −5

The 1st and 2nd radial mode far fields are plotted dimensionally, at 50.0 m away from the exhaust plane. Amplitudes are such that the cross-wise averaged intensity at $z = l$ is 1 W/m^2.



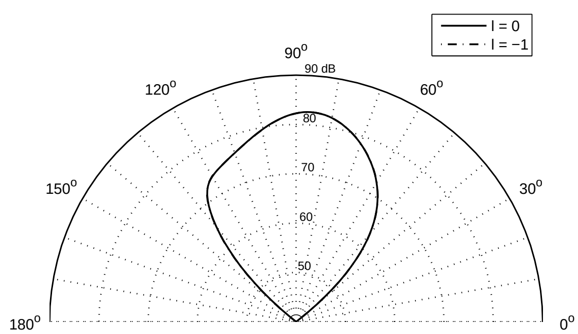
(a) 1st radial mode (b) 2nd radial mode

Figure 2: Comparing co-planar and buried exits with $m = 1$, full Kutta condition



(a) 1st radial mode (b) 2nd radial mode

Figure 3: Comparing co-planar and buried exits with $m = 10$, full Kutta condition

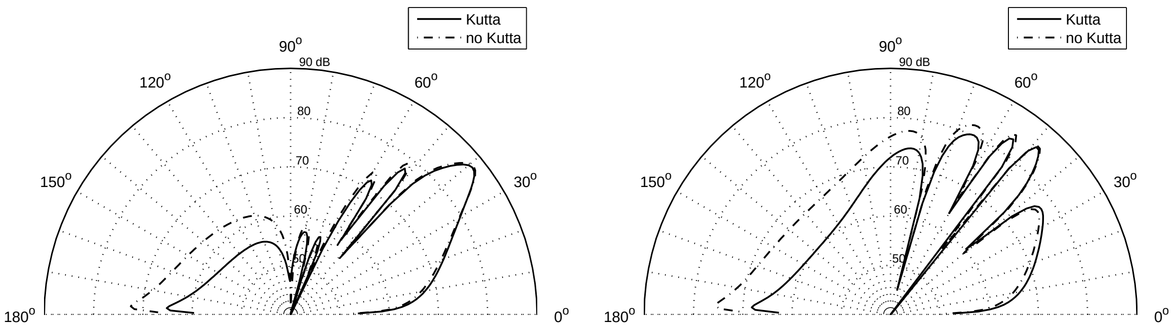


(a) 1st radial mode

Figure 4: Comparing co-planar and buried exits with $m = 18$, full Kutta condition

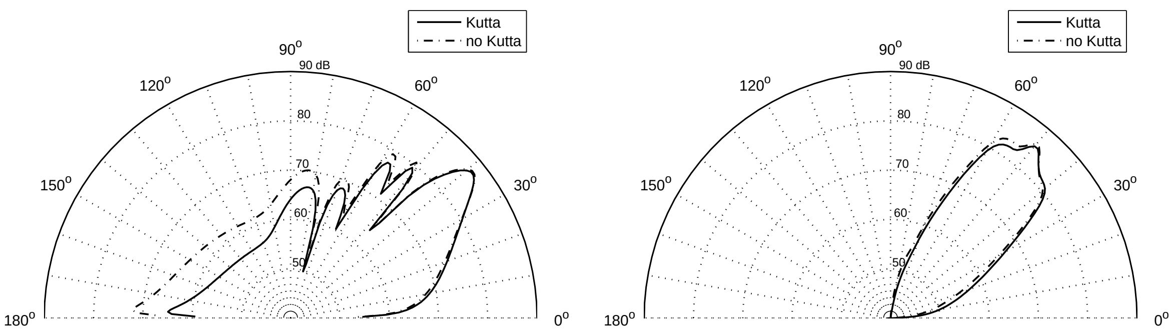
In the figures 2, 3, 4 the difference is studied of buried against co-planar nozzle. For high m the practically vanishing inner part of the modal radial profile makes the inner exit unimportant. Otherwise we see strong interference between the coplanar inner and outer exit (figure 2b).

In the figures 5,6, 7, 8, the effect is studied of the application of outer & inner edge Kutta conditions. In all cases the effect is relatively small for downstream angles and small l. This is due to the prevailing Strouhal numbers being high, which curtails the regions of significant vorticity-trailing



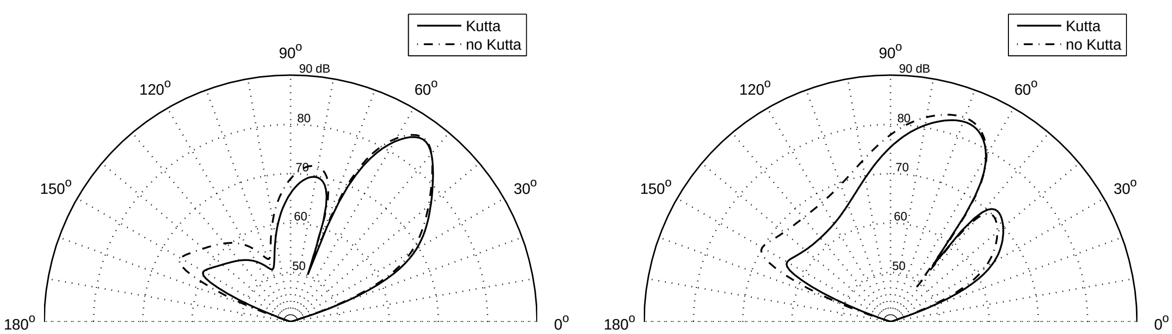
(a) 1st radial mode (b) 2nd radial mode

Figure 5: Co-planar exit with $m = 1$, comparing with and without outer&inner edge Kutta condition



(a) 1st radial mode (b) 2nd radial mode

Figure 6: Buried exit with $m = 1$, comparing with and without outer&inner edge Kutta condition

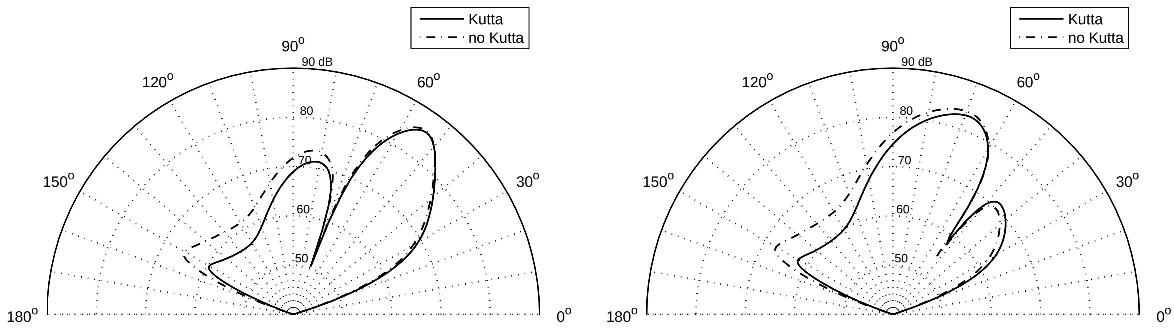


(a) 1st radial mode (b) 2nd radial mode

Figure 7: Co-planar exit with $m = 10$, comparing with and without outer&inner edge Kutta condition

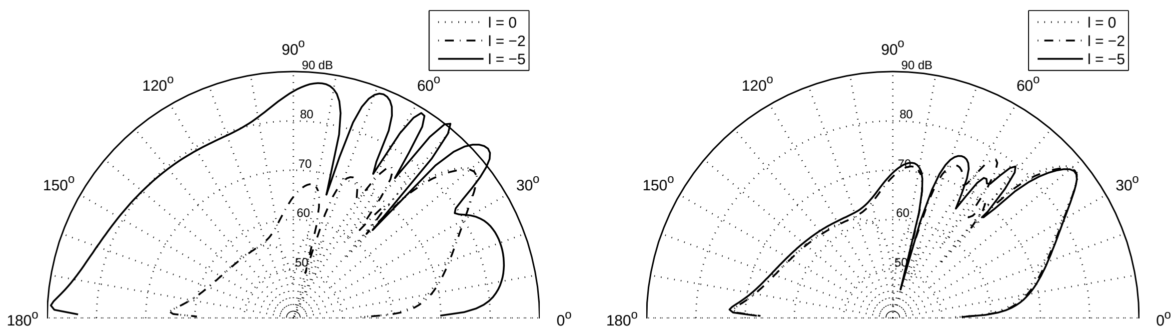
edge interaction.

When l is large enough for the hydrodynamic field, associated to the inner vortex sheet instability, to reach to the outer edge at $r = 1$, $z = 0$, the effect of inner Kutta condition becomes very large. See figure 9. Since the instability hydrodynamic field scales exponentially on the downstream distance, the acoustic field increases linearly in l on dB-scale. (This effect was reported in the presentation,



(a) 1st radial mode (b) 2nd radial mode

Figure 8: Buried exit with $m = 10$, comparing with and without outer&inner edge Kutta conditions



(a) 1st radial mode (b) 2nd radial mode

Figure 9: Comparing buried exits with $m = 1$, with and without outer&inner edge Kutta conditions

not in the paper, by Veitch & Peake in [21].) However spectacular, this effect is almost certainly an artefact of the linearisation, at least when l is larger than (say) 3.

5 Conclusions

Analytically exact solutions are constructed and numerically implemented for the relatively complicated model problem of sound radiation from an aircraft engine exit with piecewise uniform mean flow, consisting of an inner or core jet issued from a buried or co-planar exhaust, an annular by-pass jet and a co-flow. The respective mean flows may differ in velocity, temperature and density. The solution includes the effects of Kutta condition and unstable vortex sheets. The solution method is based on a Wiener-Hopf approach, supplemented by weak factorisation. This is only possible for the buried jet configuration. For a protruding inner duct geometry other courses should be pursued, like utilising Padé approximants as proposed by Veitch & Peake [21].

References

[1] R.M. Munt. "The Interaction of Sound with a Subsonic Jet Issuing from a Semi-Infinite Cylindrical Pipe". *Journal of Fluid Mechanics*, 83(4), 609–640, 1977.

[2] B. Noble, *Methods based on the Wiener-Hopf Technique*, Pergamon Press, London, 1958.

[3] H. Levine, J. Schwinger, "On the radiation of sound from an unflanged circular pipe", *Physical Review*, 73, 383–406, 1948.

[4] L.A. Weinstein, *The Theory of Diffraction and the Factorization Method*, Boulder, Colorado: Golem Press, 1969.

[5] G.F. Carrier, "Sound transmission from a tube with flow", *Quarterly of Applied Mathematics*, 13, 457–461, 1956.

[6] G.F. Homicz, J.A. Lordi, "A Note on the Radiative Directivity Patterns of Duct Acoustic Modes", *Journal of Sound and Vibration*, 41, 283–290, 1975.

[7] R.M. Munt, "Acoustic radiation from a circular cylinder in a subsonic stream", *Journal of the Institute of Mathematics and Applications*, 16, 1–10, 1975.

[8] D.S. Jones, J.D. Morgan, "The Instability of a Vortex Sheet Subsonic Stream under Acoustic Radiation", *Proceeding of the Cambridge Philosophical Society*, 72, 465–488, 1972.

[9] D.W. Bechert, "Sound Absorption Caused by Vorticity Shedding, Demonstrated with a Jet Flow", AIAA 79-0575, AIAA 5th Aeroacoustics Conference, Mar. 1979

[10] M.S. Howe, "Attenuation of Sound in a Low Mach Number Nozzle Flow", *Journal of Fluid Mechanics*, 91, 209–229, 1979.

[11] S.W. Rienstra, "A small Strouhal number analysis for acoustic wave-jet flow-pipe interaction", *Journal of Sound and Vibration* 86, 539–556, 1983.

[12] A.M. Cargill, "Low frequency acoustic radiation from a jet pipe - a second order theory", *Journal of Sound and Vibration*, 83, 339–354, 1982.

[13] S.W. Rienstra, "On the Acoustical Implications of Vortex Shedding from an Exhaust Pipe", *Journal of Engineering for Industry, Transactions of the ASME*, 103, 378–384, 1981.

[14] S.W. Rienstra, "Acoustic Radiation From A Semi-Infinite Annular Duct In A Uniform Subsonic Mean Flow", *Journal of Sound Vibration*, 94(2), 267–288, 1984

[15] S.W. Rienstra, "Sound diffraction at a trailing edge", *Journal of Fluid Mechanics*, 108, 443–460, l981.

[16] G. Gabard, R.J. Astley, "Theoretical Models for Sound Radiation from Annular Jet Pipes: Far- and Near-Field Solutions", *Journal of Fluid Mechanics*, 549, 315–342, 2006

[17] A. Demir, S.W. Rienstra, "Sound Radiation from an Annular Duct with Jet Flow and a Lined Centerbody", AIAA 2006-2718, 12th AIAA/CEAS Aeroacoustics Conference, Cambridge, MA, USA, 8-10 May 2006.

[18] M.V. Taylor, D.G. Crighton, A.M. Cargill, "The Low Frequency Aeroacoustics of Buried Nozzle Systems", *Journal of Sound Vibration*, 163(3), 493–526, 1993

[19] M. Idemen, "A New Method to Obtain Exact Solutions of Vector Wiener-Hopf Equations", *Zeitschrift für Angewandte Mathematik und Mechanik ZAMM*, 59(11), 656–658, 1979

[20] A. Büyükaksoy, A. Demir, "Radiation of Sound from a Semi-Infinite Rigid Duct Inserted Axially into a Larger Infinite Tube with Wall Impedance Discontinuity", *Journal of Applied Mathematics and Mechanics ZAMM* 86(7), 563–571, 2006

[21] B. Veitch, N. Peake, "Models for Acoustic Propagation through Turbofan Exhaust Flows", AIAA 2007-3543, 13th AIAA/CEAS Aeroacoustics Conference, Rome, 21-23 May 2007.

[22] A. Demir, S.W. Rienstra, "Sound Radiation from a Buried Nozzle with Jet and Bypass Flow", paper no. 449 at ICSV14, the Fourteenth International Congress on Sound and Vibration, Cairns, July 9-12, 2007

[23] G. Gabard, B.J. Tester, L. De Mercato, "Acoustic Near- to Far-field Characteristics of Nozzle Radiation Through Plug Flow Jets", AIAA 2007-3652, 13th AIAA/CEAS Aeroacoustics Conference, Rome, 21-23 May 2007.