4.1. *Slowly varying modes*

$$\Omega^2\Big(\frac{\Psi_y}{\Omega^2}\Big)_y + (\Omega^2-\kappa^2)\Psi = \mathrm{i}\varepsilon\Omega^2\Bigg[\frac{C}{\Omega}\Big(\frac{U\Psi}{C^2}+DA\Big)_X + \frac{\kappa D}{\Omega^2}\Big(\frac{\Psi}{D}+UA\Big)_X + \frac{1}{\Omega C}(V\Psi)_y + \frac{\kappa DV}{\Omega^2}A_y + \mathrm{i}D\Big(\frac{1}{\Omega^2}\left[UB_X+(VB)_y\right]\Big)_y\Bigg] + O(\varepsilon^2) \quad (4.5)$$

To leading order equation (4.5) reduces to the Pridmore-Brown eigenvalue equation

$$\Omega^2\Big(\frac{\Psi_{0y}}{\Omega^2}\Big)_y + (\Omega^2-\kappa^2)\Psi_0 = 0, \qquad (4.8a)$$

with boundary conditions

$$\Psi_{0y} = \frac{D\Omega^2C^2}{\mathrm{i}\omega Z_h}\Psi_0, \qquad \Psi_{0y} = -\frac{D\Omega^2C^2}{\mathrm{i}\omega Z_g}\Psi_0, \qquad (4.8b)$$

4.2. *Final solution part 1: the general case*

$$\Big[\frac{ZU}{\Omega^2C^2}\Big(\frac{\Psi_0\Psi_{0y}}{ZD\Omega C}\Big)_X + \frac{ZV}{\Omega^2C^2}\Big(\frac{\Psi_0\Psi_{0y}}{ZD\Omega C}\Big)_y + \frac{V\kappa\Psi_0^2}{UD\Omega^2C^2}\Big]_g^h = -\int_g^h \frac{\Psi_0}{DC^2}\Bigg[\frac{C}{\Omega}\Big(\Big(\kappa+\frac{U\Omega}{C}\Big)\frac{\Psi_0}{\Omega C} - \frac{U_y\Psi_{0y}}{\Omega^2C^2}\Big)_X + \frac{\kappa^2V}{\Omega^3C}\Psi_{0y} + \frac{\kappa D}{\Omega^2}\Big(\frac{\omega\Psi_0}{D\Omega C} - \frac{UU_y\Psi_{0y}}{D\Omega^2C^2}\Big)_X + \Big(\frac{V\Psi_0}{\Omega C} - \frac{DU}{\Omega^2}\Big(\frac{\Psi_{0y}}{D\Omega C}\Big)_X - \frac{1}{\Omega^2C}\Big(\frac{V\Psi_{0y}}{\Omega}\Big)_y\Big)_y\Bigg]\mathrm{d}y. \quad (4.11)$$

After partial integration we get

$$\Big[\frac{ZU}{\Omega^2C^2}\Big(\frac{\Psi_0\Psi_{0y}}{ZD\Omega C}\Big)_X + \frac{ZV}{\Omega^2C^2}\Big(\frac{\Psi_0\Psi_{0y}}{ZD\Omega C}\Big)_y + \frac{V\kappa\Psi_0^2}{DU\Omega^2C^2}\Big]_g^h + \Big[\frac{\Psi_0}{DC^2}\Big(\frac{V\Psi_0}{\Omega C} - \frac{DU}{\Omega^2}\Big(\frac{\Psi_{0y}}{D\Omega C}\Big)_X - \frac{1}{\Omega^2C}\Big(\frac{V\Psi_{0y}}{\Omega}\Big)_y\Big)\Big]_g^h = -\int_g^h \frac{\Psi_0}{DC^2}\Bigg[\frac{C}{\Omega}\Big(\Big(\kappa+\frac{U\Omega}{C}\Big)\frac{\Psi_0}{\Omega C} - \frac{U_y\Psi_{0y}}{\Omega^2C^2}\Big)_X + \frac{\kappa^2V}{\Omega^3C}\Psi_{0y} + \frac{\kappa D}{\Omega^2}\Big(\frac{\omega\Psi_0}{D\Omega C} - \frac{UU_y\Psi_{0y}}{D\Omega^2C^2}\Big)_X\Bigg] - \frac{\Psi_{0y}}{DC^2}\Big(\frac{V\Psi_0}{\Omega C} - \frac{DU}{\Omega^2}\Big(\frac{\Psi_{0y}}{D\Omega C}\Big)_X - \frac{1}{\Omega^2C}\Big(\frac{V\Psi_{0y}}{\Omega}\Big)_y\Big)\mathrm{d}y$$

By using the defining differential equation we get

$$\Big[\frac{Z\Psi_{0y}U}{D\Omega^3C^3}\Big(\frac{\Psi_0}{Z}\Big)_X + \frac{Z\Psi_{0y}V}{D\Omega^3C^3}\Big(\frac{\Psi_0}{Z}\Big)_y + \frac{V\Psi_{0y}^2}{D\Omega^3C^3} - \frac{V_y\Psi_0\Psi_{0y}}{D\Omega^3C^3} + \frac{V\Psi_0^2}{D\Omega C^3} + \frac{V\kappa\Psi_0^2}{DU\Omega^2C^2}\Big]_g^h = \int_g^h -\frac{\Psi_0}{D\Omega C}\Big(\Big(\kappa+\frac{U\Omega}{C}\Big)\frac{\Psi_0}{\Omega C}\Big)_X + \frac{\Psi_0}{D\Omega C}\Big(\frac{U_y\Psi_{0y}}{\Omega^2C^2}\Big)_X - \frac{\kappa\Psi_0}{\Omega^2C^2}\Big(\frac{\omega\Psi_0}{D\Omega C}\Big)_X + \frac{\kappa\Psi_0}{\Omega^2C^2}\Big(\frac{UU_y\Psi_{0y}}{D\Omega^2C^2}\Big)_X - \frac{U\Psi_{0y}}{\Omega^2C^2}\Big(\frac{\Psi_{0y}}{D\Omega C}\Big)_X\mathrm{d}y$$

After introducing the derivative of $(\kappa + U\Omega/C)\Psi_0^2/D\Omega^2C^2 - \omega U_y\Psi_0\Psi_{0y}/D\Omega^4C^4$, and applying partial integration we obtain

$$\left[\frac{Z\Psi_{0y}U}{D\Omega^3C^3}\left(\frac{\Psi_0}{Z}\right)_X + \frac{Z\Psi_{0y}V}{D\Omega^3C^3}\left(\frac{\Psi_0}{Z}\right)_y + \frac{V\Psi_{0y}^2}{D\Omega^3C^3} - \frac{V_y\Psi_0\Psi_{0y}}{D\Omega^3C^3} + \frac{V\Psi_0^2}{D\Omega C^3} + \frac{V\kappa\Psi_0^2}{UD\Omega^2C^2} - \left(\kappa+\frac{U\Omega}{C}\right)\frac{V\Psi_0^2}{UD\Omega^2C^2} + \frac{\omega VU_y\Psi_0\Psi_{0y}}{UD\Omega^4C^4}\right]_g^h$$

$$+\frac{\mathrm{d}}{\mathrm{d}X}\int_g^h \left(\kappa+\frac{U\Omega}{C}\right)\frac{\Psi_0^2}{D\Omega^2C^2} - \frac{\omega U_y\Psi_0\Psi_{0y}}{D\Omega^4C^4}\,\mathrm{d}y = \int_g^h \left(\left(\kappa+\frac{U\Omega}{C}\right)\frac{\Psi_0^2}{D\Omega^2C^2} - \frac{\omega U_y\Psi_0\Psi_{0y}}{D\Omega^4C^4}\right)_X - \frac{U\Psi_{0y}}{\Omega^2C^2}\left(\frac{\Psi_{0y}}{D\Omega C}\right)_X - \frac{\Psi_0}{D\Omega C}\left(\left(\kappa+\frac{U\Omega}{C}\right)\frac{\Psi_0}{\Omega C} - \frac{U_y\Psi_{0y}}{\Omega^2C^2}\right)_X - \frac{\kappa\Psi_0}{\Omega^2C^2}\left(\frac{\omega\Psi_0}{D\Omega C} - \frac{UU_y\Psi_{0y}}{D\Omega^2C^2}\right)_X \mathrm{d}y$$

Use the differential equation and $\omega = \Omega C + \kappa U$

$$\left[\frac{Z\Psi_{0y}U}{D\Omega^3C^3}\left(\frac{\Psi_0}{Z}\right)_X + \frac{Z\Psi_{0y}V}{D\Omega^3C^3}\left(\frac{\Psi_0}{Z}\right)_y + \frac{V\Psi_{0y}^2}{D\Omega^3C^3} - \frac{V_y\Psi_0\Psi_{0y}}{D\Omega^3C^3} + \frac{\omega VU_y\Psi_0\Psi_{0y}}{UD\Omega^4C^4}\right]_g^h + \frac{\mathrm{d}}{\mathrm{d}X}\int_g^h \left(\kappa+\frac{U\Omega}{C}\right)\frac{\Psi_0^2}{D\Omega^2C^2} - \frac{\omega U_y\Psi_0\Psi_{0y}}{D\Omega^4C^4}\,\mathrm{d}y =$$

$$\int_g^h -\left(\frac{U\Psi_{0y}}{\Omega^2C^2}\right)_y\left(\frac{\Psi_0}{D\Omega C}\right)_X - \frac{U\Psi_{0y}}{\Omega^2C^2}\left(\frac{\Psi_{0y}}{D\Omega C}\right)_X - \frac{\Psi_0}{D\Omega C}\left(\frac{\kappa UU_y\Psi_{0y}}{\Omega^3C^3}\right)_X + \frac{\kappa U_y\Psi_0}{D\Omega^2C^2}\left(\frac{U\Psi_{0y}}{\Omega^2C^2}\right)_X - \frac{\kappa UU_y\Psi_{0y}}{\Omega^3C^3}\left(\frac{\Psi_0}{D\Omega C}\right)_X + \frac{\kappa U\Psi_0\Psi_{0y}}{\Omega^4C^4}\left(\frac{U_y}{D}\right)_X \mathrm{d}y$$

$$\left[\frac{Z\Psi_{0y}U}{D\Omega^3C^3}\left(\frac{\Psi_0}{Z}\right)_X + \frac{Z\Psi_{0y}V}{D\Omega^3C^3}\left(\frac{\Psi_0}{Z}\right)_y + \frac{V\Psi_{0y}^2}{D\Omega^3C^3} - \frac{V_y\Psi_0\Psi_{0y}}{D\Omega^3C^3} + \frac{\omega VU_y\Psi_0\Psi_{0y}}{UD\Omega^4C^4}\right]_g^h + \frac{\mathrm{d}}{\mathrm{d}X}\int_g^h \left(\kappa+\frac{U\Omega}{C}\right)\frac{\Psi_0^2}{D\Omega^2C^2} - \frac{\omega U_y\Psi_0\Psi_{0y}}{D\Omega^4C^4}\,\mathrm{d}y =$$

$$\int_g^h -\left(\left(\frac{\Psi_0}{D\Omega C}\right)_X\frac{U\Psi_{0y}}{\Omega^2C^2}\right)_y + \frac{\kappa U\Psi_0\Psi_{0y}}{\Omega^4C^4}\left(\frac{U_y}{D}\right)_X \mathrm{d}y$$

Integrate and use $(DU)_X + DV_y = 0$

$$\left[\frac{Z\Psi_{0y}U}{D\Omega^3C^3}\left(\frac{\Psi_0}{Z}\right)_X + \frac{Z\Psi_{0y}V}{D\Omega^3C^3}\left(\frac{\Psi_0}{Z}\right)_y + \frac{V\Psi_{0y}^2}{D\Omega^3C^3} + \frac{(DU)_X\Psi_0\Psi_{0y}}{D^2\Omega^3C^3} + \frac{\omega VU_y\Psi_0\Psi_{0y}}{UD\Omega^4C^4} + \left(\frac{\Psi_0}{D\Omega C}\right)_X\frac{U\Psi_{0y}}{\Omega^2C^2}\right]_g^h + \frac{\mathrm{d}}{\mathrm{d}X}\int_g^h \left(\kappa+\frac{U\Omega}{C}\right)\frac{\Psi_0^2}{D\Omega^2C^2} - \frac{\omega U_y\Psi_0\Psi_{0y}}{D\Omega^4C^4}\,\mathrm{d}y = \int_g^h \frac{\kappa U\Psi_0\Psi_{0y}}{\Omega^4C^4}\left(\frac{U_y}{D}\right)_X \mathrm{d}y$$

$$\left[\frac{Z\Psi_{0y}}{D\Omega^2C^2\Psi_0}\left[\frac{U\Psi_0}{\Omega C}\left(\frac{\Psi_0}{Z}\right)_X + \frac{V\Psi_0}{\Omega C}\left(\frac{\Psi_0}{Z}\right)_y + \frac{V\Psi_0}{Z}\left(\frac{\Psi_0}{\Omega C}\right)_y + \frac{\Psi_0}{Z}\left(\frac{U\Psi_0}{\Omega C}\right)_X + \frac{VU_y\Psi_0^2}{ZU\Omega C}\right]\right]_g^h + \frac{\mathrm{d}}{\mathrm{d}X}\int_g^h \left(\kappa+\frac{U\Omega}{C}\right)\frac{\Psi_0^2}{D\Omega^2C^2} - \frac{\omega U_y\Psi_0\Psi_{0y}}{D\Omega^4C^4}\,\mathrm{d}y = \int_g^h \frac{\kappa U\Psi_0\Psi_{0y}}{\Omega^4C^4}\left(\frac{U_y}{D}\right)_X \mathrm{d}y$$

$$
\left[\frac{Z\Psi_{0y}}{D\Omega^2C^2\Psi_0}\left(\frac{U\Psi_0^2}{Z\Omega C}\right)_X + \frac{Z\Psi_{0y}}{D\Omega^2C^2\Psi_0}\frac{V}{U}\left(\frac{U\Psi_0^2}{Z\Omega C}\right)_y\right]_g^h
+ \frac{\mathrm{d}}{\mathrm{d}X}\int_g^h \left(\kappa + \frac{U\Omega}{C}\right)\frac{\Psi_0^2}{D\Omega^2C^2} - \frac{\omega U_y\Psi_0\Psi_{0y}}{D\Omega^4C^4}\,\mathrm{d}y = \int_g^h \frac{\kappa U\Psi_0\Psi_{0y}}{\Omega^4C^4}\left(\frac{U_y}{D}\right)_X \mathrm{d}y. \quad (4.12)
$$

With boundary conditions (4.8*b*) this becomes

$$
\left[\left(\frac{U\Psi_0^2}{\mathrm{i}\omega Z_h\Omega C}\right)_X + h_X\left(\frac{U\Psi_0^2}{\mathrm{i}\omega Z_h\Omega C}\right)_y\right]_h + \left[\left(\frac{U\Psi_0^2}{\mathrm{i}\omega Z_g\Omega C}\right)_X + g_X\left(\frac{U\Psi_0^2}{\mathrm{i}\omega Z_g\Omega C}\right)_y\right]_g
+ \frac{\mathrm{d}}{\mathrm{d}X}\int_g^h \left(\kappa + \frac{U\Omega}{C}\right)\frac{\Psi_0^2}{D\Omega^2C^2} - \frac{\omega U_y\Psi_0\Psi_{0y}}{D\Omega^4C^4}\,\mathrm{d}y = \int_g^h \frac{\kappa U\Psi_0\Psi_{0y}}{\Omega^4C^4}\left(\frac{U_y}{D}\right)_X \mathrm{d}y
$$

$$
\frac{\mathrm{d}}{\mathrm{d}X}\left[\frac{1}{\mathrm{i}\omega Z_h}\frac{U\Psi_0^2}{\Omega C}\bigg|_h + \frac{1}{\mathrm{i}\omega Z_g}\frac{U\Psi_0^2}{\Omega C}\bigg|_g\right]
+ \frac{\mathrm{d}}{\mathrm{d}X}\int_g^h \left(\kappa + \frac{U\Omega}{C}\right)\frac{\Psi_0^2}{D\Omega^2C^2} - \frac{\omega U_y\Psi_0\Psi_{0y}}{D\Omega^4C^4}\,\mathrm{d}y = \int_g^h \frac{\kappa U\Psi_0\Psi_{0y}}{\Omega^4C^4}\left(\frac{U_y}{D}\right)_X \mathrm{d}y \quad (4.13a)
$$