

# Analytical Approximations For Offshore Pipelaying Problems

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## 1. Abstract

The geometrically non-linear slender bar equation is solved for a number of problems involving suspended pipelines, related to the off-shore gas- and oil-pipe laying. The problems concern the use of a lay-barge with stinger, and the process of abandoning and recovery of a pipe. The usually stiff equation requires for a completely numerical solution considerable computer power, not always available on board. Therefore, the solutions are analytical (matched asymptotic expansions, and linear theory) to allow the results being evaluated on a small computer. It is shown that for the majority of the practical cases the two solution methods complement each other very well.

## 2. Introduction

Exploitation of gas- and oil-wells offshore requires the presence of pipelines along the sea floor for transport of the products. The laying of these pipelines is usually done by suspending the pipeline via a stinger from a lay-barge. On board the pipe is composed by welding pipe elements together at the welding ramp (figure 1). During the process of laying the pipe is bent by its own weight into a stretched  $S$ -curve, causing bending stresses in the pipe. If the water is (relatively) shallow, the pipe sufficiently stiff, and the weight (per unit of length) is sufficiently low, these stresses remain low enough without further precautions. However, in modern applications the pipes are laid in deep water, sometimes in a considerable current necessitating a heavy pipe (especially when it is a gas pipe), in a way that the bending stresses become so high that the pipe would buckle. In that case, a

horizontal tension is applied by the ship, to stretch the bends and reduce the stresses. Furthermore, sometimes the pipe has to be abandoned and recovered by means of a cable (let down to and pulled up from the sea floor, figure 2, for example when a storm prohibits the continuation of laying). Also during this process a certain tension is to be applied at the cable to avoid buckling.

Since both the tension machines, stinger equipment etc., and the repairing of a buckled pipe are very expensive, it is necessary to calculate in advance, for a given configuration, the tension just sufficient to obtain a given maximum stress level. This problem will be discussed here.

With sea current, dynamics of the sea, nonlinearity of the steel elasticity, and variation of pipe weight and flexural rigidity being usually of minor importance, we consider the model of a linearly elastic, geometrically nonlinear suspended bar, loaded by its own weight and a horizontal tension. The equations and boundary conditions will be presented in the next section; the derivation may be found in [1]. The differential equation is of second order, with an unknown free pipe length and an unknown bottom reaction. Effectively, the problem is therefore of fourth order. Due to the nonlinear character of the problem only very few exact solutions are known. For example, if the flexural rigidity vanishes we obtain the catenary (some boundary conditions have to be given up), and if the pipe weight vanishes the equation becomes equivalent to the nonlinear pendulum equation allowing an implicit solution with elliptic functions (Kirchoff's analogy, [1]). For the present problem no exact solutions are known, and we will therefore consider approximations.

A very well-known approximation is based on the pipe being nearly horizontal (small deflection angle) allowing linearized equations with solutions of exponential or (if the horizontal tension vanishes) polynomial type (beam theory; [1,2]). A generalization of this approach is a linearization around a non-zero mean deflection angle, yielding solutions in terms of Airy functions [1]. These linearizations are uniform approximations: physically, the behaviour of the pipe is everywhere the same with basically the same equation valid. Another approach is based on a small (relative) flexural rigidity, giving the pipe a shape close to a catenary (important weight and tension, unimportant bending stiffness), except for the regions near the ends where bending stiffness is important, but weight is unimportant. Evidently, this approximation is not uniform, with physically different behaviour in boundary layers at the ends. It has become known in the literature as "stiffened catenary". The presence of these boundary layers was first noted by Plunkett [3] (for a related problem without free boundaries); however, his asymptotic solution is only correct to leading order. Matched asymptotic expansion solutions based on this boundary layer behaviour were also given

by Dixon and Rutledge [4] (using Plunketts solution), van der Heyden [5] (for suspended cables), Konuk [6], and (in a more general setting) by Flaherty and O'Malley [7]. None of these authors, however, do consider the present *free* boundary problem.

In the following sections we will derive two approximate solutions (or in any case, reduce the problem to an algebraic equation): a small-angle linearization (beam), and a matched asymptotic expansion based on a small flexural rigidity (stiffened catenary). Special attention will be paid to the use of the first integral of the equation (free bending energy) to deal with the inherent problem of the unknown suspended pipe length. Furthermore, we will show that these two approximations appear to provide excellent (complementary) solutions for almost all the investigated practical cases, so that they are probably sufficient in a practical situation for on-board calculations with only a small computer available. At the same time, of course, they provide efficient starting values for completely numerical solutions which might otherwise suffer from the stiffness of the equation.

### 3. The Problems

Equilibrium of forces, together with application of the Bernoulli-Euler law, relating bending moment to radius of curvature, yields the following equation for  $\psi(s)$ , the angle between horizon and the tangent at the local coordinate  $s$ , in non-dimensional form

$$(\epsilon/\mu)^2 \psi_{ss} = \sin(\psi) - (\mu s - \lambda) \cos(\psi) \quad (1)$$

along the interval  $[0,1]$ , and where  $\epsilon^2 = EI Q^2 / H^3$ ,  $\mu = LQ/H$ ,  $\lambda = V/H$ , with  $EI$  denoting the flexural rigidity,  $Q$  the pipe weight per unit length,  $H$  the horizontal tension,  $L$  the (unknown) free pipe length, and  $V$  the (unknown) bottom reaction force. The corresponding boundary conditions for the pipelay problem are given by

$$\psi(0) = 0, \quad \psi_s(0) = 0, \quad \psi_s(1) = -\mu/r, \quad (2.a)$$

$$d = d_{sh} + r \cos(\psi(1)) - r \cos(\phi), \quad (2.b)$$

and for the abandon/recovery problem

$$\psi(0) = 0, \quad \psi_s(0) = 0, \quad \psi_s(1) = 0, \quad (3.a)$$

$$d = d_{sh} + r \cos(\gamma) - r \cos(\phi) - (c - r\gamma + r\phi) \sin(\gamma), \quad \gamma = \arctan(\mu - \lambda). \quad (3.b)$$

Here is  $r = RQ/H$ ,  $d = DQ/H$ ,  $d_{sh} = D_{sh}Q/H$ ,  $c = CQ/H$ , with  $R$  denoting the stinger radius,  $D = L \int_0^1 \sin(\psi(s)) ds$  the height of the pipe end,  $D_{sh}$  the height of the stinger hinge,  $\phi$  the angle at the hinge,  $C$  the cable length (measured from the stinger hinge), and  $\gamma$  the cable angle. The cable is for simplicity taken with zero weight, but a nonzero weight can be included without much difficulty.

An important relation is the first integral of (1), expressing the elastic free bending energy density [2], and providing an explicit relation between  $d$  and  $\psi(1)$ :

$$\frac{1}{2}(\varepsilon/r)^2 = 1 - \cos(\psi(1)) - (\mu - \lambda) \sin(\psi(1)) + d. \quad (4)$$

In the following section we will present a stiffened catenary solution for small  $\varepsilon$ , and a beam solution for small  $|\psi|$ , of (1) with (2) and (1) with (3), and using (where appropriate) (4). We note in passing that the present problems have no unique solution without an additional condition to minimize energy or pipe length; further research is in progress [8].

## 4. Solution

### 4.1 Stiffened catenary ( $\varepsilon \rightarrow 0$ )

The solution is built up from local asymptotic expansions in three regions:  $\psi = h$  in  $s = O(1)$ ,  $\psi = f$  in  $s = O(\varepsilon)$ , and  $\psi = g$  in  $s = 1 + O(\varepsilon)$ . Unknown constants are determined via matching. The complete solution is constructed by adding the three solutions and subtracting common terms:

$$\psi = \bar{f} + h + \bar{g}.$$

This whole matched asymptotic expansion procedure is relatively standard, and will not be repeated here. The only point to be noted, is that  $L$  and  $V$ , and therefore  $\mu$  and  $\lambda$ , are unknown, so dependent on  $\varepsilon$ , and should therefore be expanded into powers of  $\varepsilon$ , like  $f, g$  and  $h$ . This will, however, not be carried through right from the start. It is more convenient to begin with assuming  $\mu$  and  $\lambda$  fixed, or rather, known to any desired accuracy, and to postpone the actual calculation to a later stage.

If  $s = O(1)$ , we introduce  $z = \mu s - \lambda$ , and rewrite (1) into

$$h = \arctan(z) + \arcsin(\varepsilon^2 h_{zz} / (1+z^2)^{1/2}).$$

By successive substitution, or otherwise, we obtain easily

$$h = \arctan(z) - 2\varepsilon^2 z / (1+z^2)^{5/2} + O(\varepsilon^4). \quad (5)$$

Note that the leading order term is just the catenary.

If  $s = O(\varepsilon)$ , we introduce  $t = \mu s / \varepsilon$  to obtain

$$f_{tt} = \sin(f) - (\varepsilon t - \lambda) \cos(f).$$

Expanding  $f$  and  $\lambda$  (which is determined at this stage) in an  $\varepsilon$ -power series, yields, after matching and application of the boundary conditions,

$$\lambda = \varepsilon / (1 + \frac{3}{4}\varepsilon^2) + O(\varepsilon^5), \quad (6)$$

$$\bar{f} = \varepsilon e^{-t} - \frac{1}{48} \varepsilon^3 e^{-t} [e^{-2t} + 4t^3 - 6t^2 + 6t + 147] + O(\varepsilon^5). \quad (7)$$

(We already skipped the terms common to  $f$  and  $h$ ).

If  $s = 1 + O(\varepsilon)$ , we introduce  $\tau = \mu(s-1)/\varepsilon\kappa$ , where  $\kappa = (1 + (\mu-\lambda)^2)^{-1/4}$ , to obtain

$$g_{\tau\tau} = \kappa^2 \sin(g) - \kappa^2(\mu - \lambda + \varepsilon\kappa\tau) \cos(g).$$

Following the usual steps we arrive at

$$\bar{g} = -\varepsilon\kappa e^\tau (\kappa^4 + 1/r) [1 + \frac{1}{4}\varepsilon(\mu-\lambda)\kappa^5(\tau^2 - \tau + 1)] + O(\varepsilon^3) \quad (8)$$

for the pipelay problem; in case of the abandon/recovery problem the term  $1/r$  is set to zero. Up to now we have applied the boundary conditions (2.a) and (3.a). The final step to be taken, to determine  $\mu$ , is substitution of the results obtained so far into (2.b) and (3.b), and utilizing (4) to get rid of  $d$ . Rewritten in suitable form it becomes for the pipelay problem

$$\kappa^2 = [A + (A^2 + 4r(\cos \alpha - (\mu-\lambda)\sin \alpha)\cos \alpha)^{1/2}] / 2r(\cos \alpha - (\mu-\lambda)\sin \alpha) \quad (9)$$

with  $A = r \cos(\phi) - d_{sh} + \frac{1}{2}(\varepsilon/r)^2 - 1$ , and  $\alpha = \psi(1) - \arctan(\mu-\lambda)$ . Since  $\alpha = O(\varepsilon)$ , the right-hand side of (9) is to leading order independent of  $\varepsilon$ , and the solution for  $\mu$  is simply obtained by successive substitution of  $\kappa^2$ , starting with  $\alpha = 0$ . The equation for the abandon/recovery problem,

corresponding to (9), is

$$r\kappa^2 - (c+r\phi-r\gamma)(\mu-\lambda)\kappa^2 - \cos(\alpha)/\kappa^2 = A, \quad (10)$$

but this equation cannot be written in a form allowing an explicit asymptotic solution, and therefore has to be solved numerically.

This solves the present stiffened catenary problems. One final remark to be made is that it is practically very useful to modify the boundary layer contributions  $\bar{f}$  and  $\bar{g}$  a little bit, by adding exponentially small terms, of the order of  $\exp(-\mu/\epsilon)$  and  $\exp(-\mu/\epsilon\kappa)$ , in a way that the coupling between  $\bar{f}$  and  $\bar{g}$  in each others domain is reduced, for example:

$\psi = \bar{f}(s) - \bar{f}(1)s + h(s) + \bar{g}(s) - \bar{g}(0)(1-s)$ , and similarly for  $\psi_s$ . Asymptotically for  $\epsilon \rightarrow 0$ , these terms have no meaning, of course, since they are smaller than any power of  $\epsilon$ , but for any finite  $\epsilon$  they appear to be very useful, and extend the region of validity to values of  $\epsilon$  as high as 0.35 .

4.2 *Beam* ( $|\psi| \ll 1$ ).

Linearization of equation (1) yields

$$(\epsilon/\mu)^2 \psi_{ss} = \psi - \mu s + \lambda, \quad (11)$$

with solution (satisfying  $\psi(0)=\psi_s(0)=0$ )

$$\psi = \lambda(\cosh(t)-1) - \epsilon(\sinh(t)-t) \quad (12)$$

where  $t = \mu s/\epsilon$ . From (12) we derive an expression for  $d = \mu \int_0^1 \psi ds$  by direct integration (eq. (4) is not a first integral of (11) any more). Then, for given  $d$ , the solution  $x=x_0$  of

$$\frac{1}{2}x \sinh(x) - \cosh(x) + 1 - (d/\epsilon^2)\sinh(x)/x - (\sinh(x)/x - 1)/r = 0 \quad (13)$$

gives  $\mu = \epsilon x_0$ ,  $\lambda = \mu/2 - d/\mu - \epsilon^2/\mu r$ , of course with  $1/r = 0$  in case of the abandon/recovery problem. Finally,  $d$  is determined by solving equation (2.b) or (3.b). So for the beam problem we arrive at two coupled algebraic equations. We note, that we do *not* linearize (2.b) (which would imply  $\cos(\psi(1)) = 1$ ), since in that case we would lose all information on the lift-off angle  $\psi(1)$ , which is of great practical importance as it determines the required length of the stinger.

## 5. Examples

We start with an example of the pipeline problem, figure 3. We plotted the curve of required tension versus water depth (i.e.  $D_{sh}$ ) to obtain a prescribed minimum radius of curvature. The plot is given in dimensional form, since we scaled previously on  $H$ , which is now the varying quantity. We see the results from the beam theory, valid up to, say,  $20^\circ$ – $25^\circ$ , smoothly taken over by the stiffened catenary theory, suggesting correct results from both theories also in the transition region. This appeared to be typical for practically all the cases considered, and is, furthermore, confirmed by comparison with completely numerical results. For example, the results in the transition region near  $D_{sh}=50$  appear to be indeed very accurate, in spite of the rather high value of  $\epsilon=0.32$ , the boundary layer widths of 0.23 and 0.27, and the rather large maximum angle of about  $24^\circ$ , as is seen from the following comparison with a completely numerical solution at  $D_{sh}=50$  and  $H=110$ :

	numerical	catenary	beam
$\psi(1)$	21.89°	21.82°	22.18°
$V$	30.05	30.63	30.30
$L$	145.80	146.9	141.8
min.radius	208.2	209.2	202.0

This is much more accurate than could be estimated theoretically: an error of  $O(\epsilon^3)$  for  $\psi$ ,  $V$  and  $L$  in the catenary theory would predict 3% (here 0.3%), an error of  $O(\epsilon^2)$  for the minimum radius would give 10% (here 0.5%), and an error of  $O(|\psi|^2)$  in the beam theory, giving 15%, is really less than 3%. This remarkably better performance than the a-priori estimates remains also for other examples, and does not seem to be accidental. Probably, the higher order corrections are numerically small or cancel each other.

In figure 4 we have an example of the abandon/recovery problem, where maximum bending stress ( $\sim\psi_s$ ) is plotted versus cable length. A similar transition from catenary to beam is seen, with again overall accurate results.

## 6. Acknowledgments

We would like to thank C.J. Negenman for posing the problem (many years ago), and his advice and stimulation. The completely numerical results used for comparison were obtained by the use of a subroutine for nonlinear boundary value problems, developed by R.M. Mattheij and G. Staarink.

## 7. References

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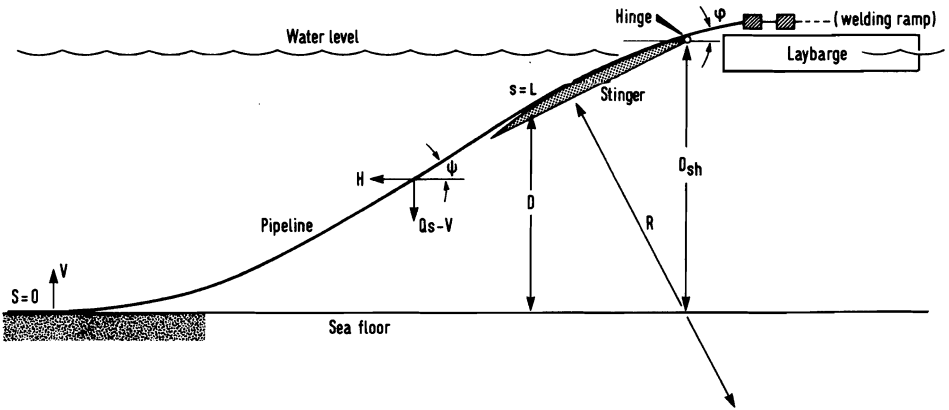


Figure 1. Sketch of the pipelay problem

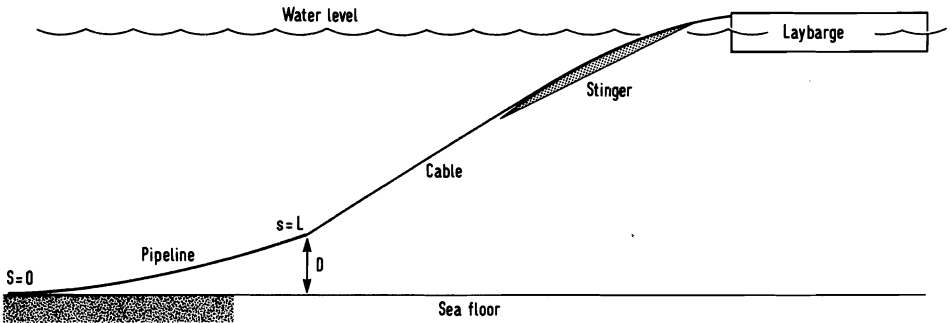


Figure 2. Sketch of the abandon / recovery problem

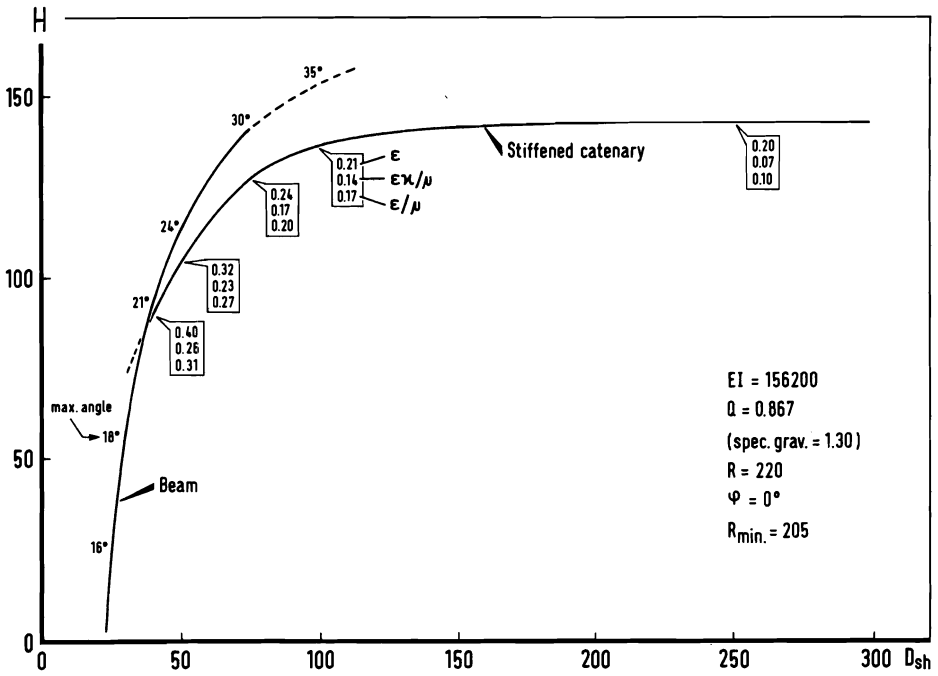


Figure 3. Tension / waterdepth diagram for constant minimum radius of curvature ( $R_{min}$ )

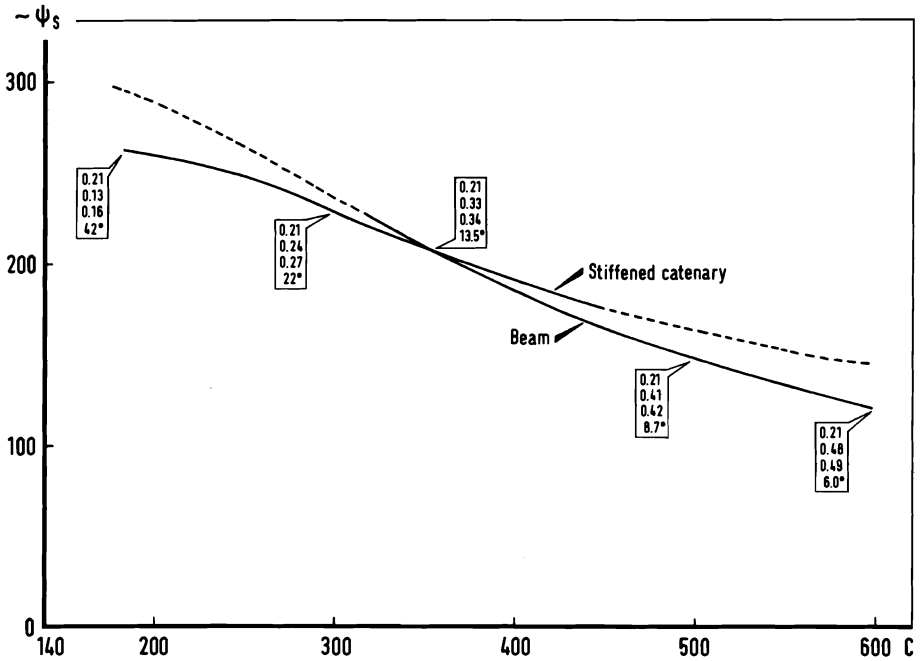


Figure 4. Maximum bending stress / cable length for pipe of fig. 3, with  $D_{sh} = 150$  and  $H = 140$ .  
Boxed numbers:  $\epsilon$ ,  $\epsilon\lambda/\mu$ ,  $\epsilon/\mu$ , maximum angle.