Journal of Engineering Mathematics **24**: 193–202, 1990.

The shape of a sessile drop for small and large surface tension

S.W. RIENSTRA

Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600MB Eindhoven, The Netherlands

Received 1 August 1989; accepted 18 December 1989

Abstract. Asymptotic solutions for large and small surface tension are developed for the profile of a symmetric sessile drop. The problem for large surface tension (i.e., small Bond number) is a regular perturbation problem, where the solution may be written as a uniformly valid asymptotic expansion. The problem for small surface tension (i.e., large Bond number) is a singular perturbation problem with boundary-layer behaviour in the edge region. The solution is a matched asymptotic expansion, where some care is to be taken for the matching. The respective ranges of validity are established by comparing the asymptotic results with solutions obtained by numerical integration of the full equations.

1. Introduction

The problem of the sessile drop, that is a drop of liquid at rest on a horizontal surface with the effect of gravity being balanced by surface tension, is classical, and numerical solutions are known for more than a century [1, 2]. So the problem is rather well understood, and results of the theory are found in many applications. For example, a technique to measure surface tension experimentally is based on comparing measured and calculated profiles of a drop of suitable size [3, 4].

In addition to numerical solutions it is always useful to have (approximate) analytical solutions available, as they provide trends, typical scalings, insight into the way various effects interact, and, if their accuracy is sufficient, a possibly more convenient alternative for obtaining numerical figures. Rather few analytical solutions are known, however. Only for the limit of high surface tension (small Bond number), in which case the drop shape is that of a perturbed sphere, perturbation-series expansions are given in [5–7], although in [8] a boundary-layer type solution for large Bond number is given for the related problem of a meniscus in a vertical cylinder.

The purpose of the present paper is to complete the description of the drop shape by presenting the solution for the limit of small surface tension (high Bond number). As the drop becomes, for vanishing surface tension, an infinitely large and infinitely thin film of liquid, it is readily seen that the limit is singular, and the problem will indeed appear to be a singular perturbation problem with boundary layers at the outer edge.

The solution will be a matched asymptotic expansion [9–12], with an 'inner' part for the boundary layer and an 'outer' part for the (practically horizontal) rest of the drop surface. The validity of the solution has been greatly improved by considering the logarithm of the vertical coordinate rather than the coordinate itself. In this way the otherwise exponentially small, and therefore asymptotically vanishing, profile description of the outerpart has been made to become within reach of the asymptotic expansion.

In addition to this high-Bond-number solution we will also briefly include, for reference, our version of a small-Bond-number solution. Then both will be compared to the 'exact' numerical solution, and we will see that the ranges of validity are almost complementary.

2. The problem

We assume the sessile drop of uniform density, on a perfectly flat horizontal surface, in a uniform outer medium with negligible density so that we can assume a constant pressure, with no other external forces than (vertically orientated) gravity. Then the shape of the drop is determined by its density ρ, the gravity acceleration g, the drop size L, the surface tension γ (which is really a force per length), and the contact angle α of the drop surface at the line of contact with the supporting surface [13]. As we will see, the profile is only indirectly dependent on α, so we can take for the moment $\alpha > 90°$ (non-wetting condition). We introduce cylindrical coordinates (r, θ, z) such that the z-axis is vertically directed *down*wards, i.e. in the direction of the gravity. As there are no circumferential forces, the geometry may be taken cylindrically symmetrical about the z-axis. Following Kuiken [4], we take for L the largest radius of the circular horizontal cross sections. Since $\alpha > 90°$, this is the radius where the drop surface is just vertical. (If the volume rather than the diameter is known, L is of course part of the solution to be determined.) The ratio between the typical weight $\rho g L^3$ and the typical force due to surface tension γL is the Bond number

$$B = \frac{\rho g L^2}{\gamma} . \tag{1}$$

This dimensionless number (also called 'shape factor') characterizes the type of drop: a small B corresponds to a dominating surface tension, with a nearly spherical geometry, and a large B corresponds to a dominating weight, with a very flat drop shape (like a pancake, or a circular sheet of liquid).

As the liquid is stationary, the fluid's pressure gradient is balanced by the gravity force

$$\nabla p = \rho \mathbf{g} \tag{2}$$

so $p = p_0 + \rho g z$. The constant p_0 is part of the solution to be determined.

The drop surface is determined by Laplace's equation, which is the condition that at the surface the pressure difference with outside is proportional, by a factor γ, to $\nabla \cdot \mathbf{n}$ (equal to the sum of the principal curvatures of the surface), where $\mathbf{n}$ denotes a vector field, (outward) normal to the surface at the surface. (Note that it can be proved that any smooth $\mathbf{n}$ yields the same $\nabla \cdot \mathbf{n}$.)

If we introduce the azimuthal tangent angle ψ of the surface, parametrized by arc length s, such that

$$\mathbf{n} = (\sin\psi, -\cos\psi) , \qquad r = \int_0^s \cos\psi \, \mathrm{d}s' , \qquad z = \int_0^s \sin\psi \, \mathrm{d}s' , \tag{3}$$

(where $z = 0$ is taken at the top of the drop), then we have

$$\nabla \cdot \mathbf{n} = \frac{\partial n_r}{\partial r} + \frac{\partial n_z}{\partial z} + \frac{n_r}{r} = \frac{\mathrm{d}\psi}{\mathrm{d}s} + \frac{\sin\psi}{r} .$$

The equation to be solved for ψ and p_0 is thus

$$p_0 + \rho g z = \gamma\left(\frac{d\psi}{ds} + \frac{\sin\psi}{r}\right), \tag{4}$$

with boundary conditions

$$\psi = 0 \text{ at } s = 0; \qquad \psi = \pi/2 \quad \text{at } r = L \quad \text{for some } s = s_\nu .$$

In some literature [13, 14] this equation is further simplified by shifting the origin to $z = -p_0/\rho g$, at the expense, however, of an unknown height of the supporting surface. We will not follow this approach here, since this height would have to become part of the solution anyway, and, furthermore, since this height is basically a pressure, we don't think its use would clarify the solution.

As ψ_s remains positive for all ψ between 0 and π (see the Appendix), ψ keeps growing and assumes any value on this interval, so the boundary condition of $\psi = \alpha$ does not determine the drop shape, and can be ignored here. Only if, in practice, the drop volume rather than L, or its equivalence if $\alpha < 90°$ (wetting condition), is given, α may become relevant indirectly. With respect to this it may be useful to recall from [13] the explicit relation between volume and α, p_0 and the profile coordinates $r = r_\alpha$, $z = z_\alpha$ at the line of contact.

By rewriting (4) into

$$zr = \frac{\gamma}{\rho g}\frac{d}{dr}(r\sin\psi) - \frac{p_0}{\rho g} r$$

we can integrate the volume integral explicitly

$$V = \int_0^L (z_+ - z_-) 2\pi r \, dr = \frac{2\pi r_\alpha}{\rho g}\left(\tfrac{1}{2}(\rho g z_\alpha + p_0) r_\alpha - \gamma \sin\alpha\right) \tag{5}$$

(valid both for $\alpha \geqslant \pi/2$ and $\alpha < \pi/2$).

Finally, the problem is reformulated in non-dimensional quantities, to make explicit the dependence on the single characterizing dimensionless number, the Bond number. This is both physically useful and necessary for the asymptotic analysis.

Introduce

$$t = s/L, \qquad \xi = r/L, \qquad \eta = z/L, \qquad \beta = p_0 L/\gamma, \qquad B = \rho g L^2/\gamma,$$

so that ψ and β are to be determined from

$$\begin{aligned} &\psi_t + \frac{\sin\psi}{\xi} = \beta + B\eta, \\ &\psi = 0 \quad \text{at } t = 0, \\ &\psi = \frac{\pi}{2} \quad \text{at } \xi = 1 \quad \text{for some } t = t_\nu. \end{aligned} \tag{6}$$

3. Small Bond number

For small Bond number the problem is a regular perturbation problem, allowing a uniform approximation. The structure of the equation suggests a power-series expansion in B. Physically evident is that we may expect ψ, ξ, η, and t_ν to be of order 1, while at the same time it is reasonable to expect that also $\beta = \mathrm{O}(1)$ since the internal pressure p_0 has to balance the surface tension. Furthermore, it is for the moment convenient to work with a fixed interval, so we rescale

$$t = t_\nu \tau\,, \qquad \xi = t_\nu \hat{\xi}\,, \qquad \eta = t_\nu \hat{\eta}\,, \qquad \beta = \hat{\beta}/t_\nu$$

to obtain

$$\psi_\tau + \sin\psi/\hat{\xi} = \hat{\beta} + Bt_\nu^2 \hat{\eta}\,, \tag{7}$$

with $\psi(0) = 0$, $\psi(1) = \pi/2$, $t_\nu = 1/\hat{\xi}(1)$.

We substitute the expansions

$$\psi = \psi_0 + B\psi_1 + \cdots, \qquad \hat{\xi} = \hat{\xi}_0 + B\hat{\xi}_1 + \cdots,$$
$$\hat{\eta} = \hat{\eta}_0 + B\hat{\eta}_1 + \cdots, \qquad \hat{\beta} = \hat{\beta}_0 + B\hat{\beta}_1 + \cdots,$$
$$t_\nu = t_0 + Bt_1 + \cdots$$

into equation (7), and collect like powers of B to obtain equations for ψ_0, ψ_1, etc. After differentiating away $\hat{\xi}_0, \hat{\xi}_1, \ldots$, and using the fact that we look for a solution regular in $\tau = 0$, these equations can be integrated, and we obtain to order $\mathrm{O}(B^2)$

$$\psi_0 = \frac{\pi}{2}\tau\,, \qquad t_0 = \frac{\pi}{2}\,, \qquad \hat{\beta}_0 = \pi\,,$$
$$\hat{\xi}_0 = \frac{2}{\pi}\sin\psi_0\,, \qquad \hat{\eta}_0 = \frac{4}{\pi}\sin^2(\tfrac{1}{2}\psi_0)$$

and

$$\psi_1 = \frac{4}{3\pi^2}\, t_\nu^2(\tau - 2\sin\psi_0 + \tan(\tfrac{1}{2}\psi_0))\,,$$
$$\hat{\beta}_1 = -\frac{4}{\pi^2}\, t_\nu^2\Big(\frac{\pi}{2} - \frac{2}{3}\Big)\,,$$
$$\hat{\xi}_1 = \frac{8}{3\pi^3}\, t_\nu^2\Big(\tau\cos\psi_0 + \Big(1 - \frac{2}{\pi} - \cos\psi_0\Big)\sin\psi_0\Big)\,,$$
$$\hat{\eta}_1 = \frac{8}{3\pi^3}\, t_\nu^2\Big(\tau\sin\psi_0 - \sin^2\psi_0 + 2\Big(1 - \frac{2}{\pi}\Big)\sin^2(\tfrac{1}{2}\psi_0) + \log(\cos^2(\tfrac{1}{2}\psi_0))\Big)\,.$$

So, to the present accuracy,

$$t_\nu = \frac{\pi}{2}\Big/\Big(1 + \frac{4}{3\pi^2}\, Bt_\nu^2\Big(1 - \frac{2}{\pi}\Big)\Big)\,.$$

Note that, with some opportunism, we have retained in the equations for the *highest* order considered (i.e., for ψ_1) the full t_ν rather than the systematic t_0. Asymptotically, this is equivalent, and for finite B it appears to improve the accuracy of the solution considerably. Only, the expression for t_ν is now an equation to be solved (numerically, in practice).

4. Large Bond number

As the parameter $B^{-1/2}$ will be seen to be fundamental, we introduce the small parameter

$$\varepsilon = B^{-1/2}$$

and hence consider the equation

$$\varepsilon^2\left(\psi_t + \frac{\sin\psi}{\xi}\right) = \varepsilon^2\beta + \eta \tag{8}$$

for $\varepsilon \to 0$. As a start, we just try $\psi_t \leq O(1)$ and $\xi = O(1)$, and find in first approximation: $\eta = -\varepsilon^2\beta$ is a constant. Further iteration will not improve on this result because $\psi \equiv 0$ if η is constant. In other words, and ε-power series expansion of ψ yields only vanishing coefficients. The value of this constant η follows from the boundary condition at $t = 0$. However, application of $\eta(0) = 0$ is not immediately possible, because the solution may have a boundary-layer structure near $t = 0$, in view of the possible singularity at $\xi = 0$. The relevant scaling, yielding a balance between all terms of the equation, is: $\psi = \delta(\varepsilon)\tilde{\psi}$, $t = \varepsilon\tilde{t}$, $\xi \simeq t$, $\eta = \varepsilon\delta(\varepsilon)\tilde{\eta}$, $\beta = \delta(\varepsilon)\varepsilon^{-1}\tilde{\beta}$, with $\delta(\varepsilon) = o(1)$ but yet unknown. Under this transformation the equation becomes for $\varepsilon \to 0$:

$$\tilde{\psi}_{\tilde{t}} + \tilde{t}^{-1}\tilde{\psi} = \tilde{\beta} + \tilde{\eta}\,, \qquad \tilde{\psi} = \tilde{\eta}_{\tilde{t}}$$

with solution

$$\tilde{\eta} = AI_0(\tilde{t}) + BK_0(\tilde{t}) - \tilde{\beta}$$

where I_0 and K_0 are modified Besselfunctions ([15]). Since $\tilde{\eta}$ is finite at $\tilde{t} = 0$, we have $B = 0$ and $\tilde{\beta} = A$, and since $\tilde{\eta}$ is finite for $\tilde{t} \to \infty$, we have $A = 0$. So the (outer) solution for $t = O(1)$ is valid down to $t = 0$, and so, with this approximation, $\beta = 0$, and thus $\eta = 0$, $\psi = 0$, $\xi = t$.

A result like this is, however, rather meagre. If ψ is asymptotically smaller than any power of ε, a linearisation for small ψ and η is obviously justified. The resulting equation

$$\varepsilon^2(\eta_{tt} + t^{-1}\eta_t) = \varepsilon^2\beta + \eta\,, \tag{9}$$

is similar to the one just considered near $t = 0$, and has the solution

$$\eta = \varepsilon^2\beta(I_0(t/\varepsilon) - 1)\,, \qquad \psi = \varepsilon\beta I_1(t/\varepsilon)\,, \qquad \xi = t\,. \tag{10}$$

Since I_0 becomes exponentially large for large values of its argument, β must be exponentially small in ε, which indeed confirms the previous result $\beta = 0$, $\eta = 0$ and $\psi = 0$. At the same time it follows that the error in the present result is exponentially small in ε, since we ignored terms quadratically small in η. The precise value of β now results from the boundary condition $\psi = \pi/2$ at $\xi = 1$. However, with $\psi = O(1)$ the linearisation is not justified and we have to consider the full equation near $\xi = 1$, i.e. near $t = t_\nu$. After introducing the scaling

$$t = t_\nu + \varepsilon\tau$$

and using the fact that ψ is exponentially small in the region $0 < t_\nu - t = O(1)$ we obtain in the region $\tau = O(1)$ the estimates

$$\eta = O(\varepsilon)\,, \qquad \xi = 1 + O(\varepsilon)\,, \qquad \psi_t = O(\varepsilon^{-1})\,.$$

These yield (with $\varepsilon \to 0$) an essentially new equation, namely a balance between height η and curvature ψ_t, and therefore suggests a boundary layer in $\tau = O(1)$ of the asymptotic solution. (In the terminology of [9] this scaling provides a significant, or rich, degenerate equation.) Consistency of the final result will then support this suggestion.

We introduce for convenience

$$\bar{\eta} = \eta + \varepsilon^2\beta\,,$$

and expand

$$\psi = \psi_0 + \varepsilon\psi_1 + \cdots\,, \qquad \bar{\eta} = \varepsilon\bar{\eta}_0 + \varepsilon^2\bar{\eta}_1 + \cdots\,,$$
$$\xi = 1 + \varepsilon\xi_0 + \varepsilon^2\xi_1 + \cdots\,, \qquad t_\nu = 1 + \varepsilon t_{\nu 1} + \varepsilon^2 t_{\nu 2} + \cdots\,.$$

After substitution in equation (8) and the boundary conditions, and collecting equal powers of ε, we obtain

$$\bar{\eta}_0 = \psi_{0\tau}\,, \qquad \bar{\eta}_{0\tau} = \sin\psi_0\,, \qquad \xi_{0\tau} = \cos\psi_0\,,$$

$$\text{with} \quad \psi_0(0) = \pi/2\,, \quad \xi_0(0) = 0\,,$$

$$\text{and} \quad \bar{\eta}_1 = \psi_{1\tau} + \sin\psi_0\,, \qquad \bar{\eta}_{1\tau} = \psi_1\cos\psi_0\,, \qquad \xi_{1\tau} = -\psi_1\sin\psi_0$$

$$\text{with} \quad \psi_1(0) = 0\,, \quad \xi_1(0) = 0\,, \quad \text{etc.}$$

Constants of integration, t_ν, and (finally) β are found from matching this inner solution for $\tau \to -\infty$ with the outer solution (10) for $t - t_\nu \to 0$.

We have to be careful, however, because the asymptotic relationship, established between inner and outer solution in the region of overlap, depends on the form we adopt for the solution. If we expand the exponentially small outer solution η or ψ of (10) in an ε-power series, the result is, as we have seen above, asymptotically identically equal to zero, providing no more information on β than that $\beta = 0$. On the other hand, for $\log\psi$, $\log\bar{\eta}$ and $\log\beta$ the outer solution ε-power series expansion is non-zero and does yield a non-trivial β. Therefore, the matching used here will be between inner and outer expansion of

$$\log\psi\,, \quad \log\bar{\eta}\,, \quad \text{and} \quad \xi\,.$$

(Note that just because of η and β both being exponentially small for $\tau \to -\infty$, we have combined η and β into $\bar{\eta}$, even though $\bar{\eta}$ is for any $\tau = O(1)$ asymptotically equivalent to η.)

An obvious consequence of this logarithmic matching is that a composite expansion of ψ or $\bar{\eta}$ (where the inner and outer solution are combined into one uniformly valid formula) should be based on multiplication of inner and outer solution, with common factors divided out, rather than addition and subtraction ([10]).

By elementary methods the equations for the inner solution can now be integrated. Using the condition that $\psi_0 \to 0$ as $\tau \to -\infty$ we obtain

$$\psi_0 = 4\,\mathrm{arctg}(\lambda\, e^{\tau})\,, \qquad \bar{\eta}_0 = \frac{4\lambda\, e^{\tau}}{1+\lambda^2 e^{2\tau}}\,, \qquad \xi_0 = \tau + 2 - \sqrt{2} - \frac{4\lambda^2 e^{2\tau}}{1+\lambda^2 e^{2\tau}}\,,$$

$$\text{with} \quad \lambda = \mathrm{tg}(\tfrac{1}{8}\pi) = \sqrt{2} - 1\,.$$

Similarly,

$$\psi_1 = Q\bar{\eta}_0\,,$$

$$\bar{\eta}_1 = \left[(1-\lambda^2 e^{2\tau})Q + \frac{1}{6}\lambda^4 e^{4\tau} + \frac{1}{2}\right]\frac{\bar{\eta}_0}{1+\lambda^2 e^{2\tau}}$$

$$\xi_1 = \frac{1}{3} + \frac{2}{3}\log\frac{1+\lambda^2 e^{2\tau}}{1+\lambda^2} - \frac{1}{3}e^{2\tau}\left(\frac{1+\lambda^2}{1+\lambda^2 e^{2\tau}}\right)^2 - \frac{1}{2}Q\bar{\eta}_0^2$$

$$\text{with} \quad Q = \frac{1}{12}\lambda^2(e^{2\tau}-1) - \frac{1}{2}\tau + \frac{2}{3}\log\frac{1+\lambda^2 e^{2\tau}}{1+\lambda^2}\,.$$

Matching ξ for $\tau \to -\infty$ with t + (*exponentially small terms*) gives

$$t_{\nu 1} = 2 - \sqrt{2}\,, \qquad t_{\nu 2} = \frac{1}{3} - \frac{2}{3}\log(1+\lambda^2)\,.$$

Matching $\log \bar{\eta}$ with $\log(\varepsilon^2 \beta I_0(t/\varepsilon))$, using the asymptotic result for I_0 ([15]),

$$I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left(1 + \frac{1}{8z} + \cdots\right) \quad (z \to \infty)\,,$$

yields

$$\beta = \frac{4\lambda\sqrt{2\pi t_\nu}}{\varepsilon\sqrt{\varepsilon}}\exp\left(-\frac{1}{\varepsilon} - 2 + \sqrt{2} + \varepsilon\left(\frac{1}{6}\sqrt{2} - \frac{5}{24}\right)\right).$$

Composite expansions can now be constructed as

$$\xi = 1 + \varepsilon\xi_0 + \varepsilon^2\xi_1\,,$$

$$\bar{\eta} = \frac{(\bar{\eta}_0 + \varepsilon\bar{\eta}_1)\varepsilon^2\beta I_0(t/\varepsilon)}{4\lambda\exp\left[\left(1 - \frac{1}{2}\varepsilon\right)\tau + \varepsilon\left(\frac{1}{2} - \frac{1}{12}\lambda^2 - \frac{2}{3}\log(1+\lambda^2)\right)\right]}\,.$$

They may be further refined to

$$\xi := (\xi - \xi(0))/(1 - \xi(0)) ,$$
$$\eta := (\varepsilon^2\beta/\bar{\eta}(0))^{1-t/t_v}\bar{\eta} - \varepsilon^2\beta .$$

The differences are asymptotically negligible but practically significant.

Finally, we note that by substituting the inner solution $\varepsilon\eta_0$ at $\psi = \alpha$ in expression (5) for the volume V, we may derive the following simple relation (to leading order) between the volume V, radius L, surface tension γ, and contact angle α

$$V = 2\pi L^3 \varepsilon \sin(\tfrac{1}{2}\alpha) .$$

In fact, if V and γ are given, measuring L may be an easy way to determine α.

5. Comparison with numerical solution

In order to assess the accuracy of the present solutions, plots of β for $B \to 0$ and $B \to \infty$ are given in Figs. 1 and 2, together with the 'exact' (numerically evaluated) solution. The numerical scheme used is a Fehlberg fourth-fifth order Runge–Kutta method [16], starting with a Taylor series solution near $t = 0$. The accuracy for $B \to 0$ is seen to be excellent for $B < 0.5$, and good up to $B = 2$; for $B \to \infty$ the accuracy is excellent beyond $B = 10$, and good down to $B = 4$. Note that the asymptotic solution for $B \to \infty$ near $B = 2$ rather abruptly jumps away from the exact solution, whereas the solution for $B \to 0$ is in general less accurate but on the other hand remains to be longer of the right order of magnitude.

Furthermore, some examples of drop profiles for $B \to 0$ and $B \to \infty$ are displayed in Figs. 3 and 4, again together with the numerical solution. The accuracy is seen to vary with B in a similar way as for β.

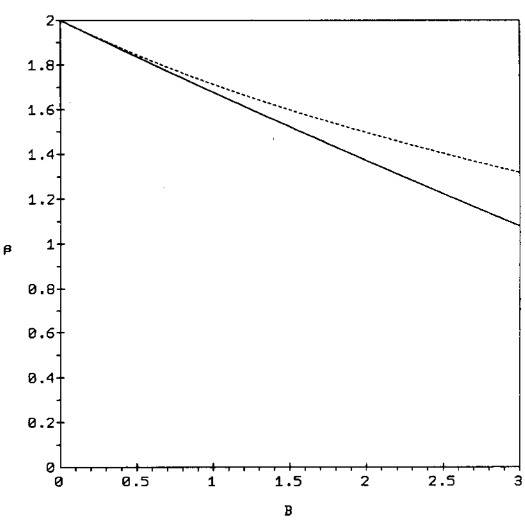

Fig. 1. β for B small. —— asymptotic solution; ------ numerical solution.

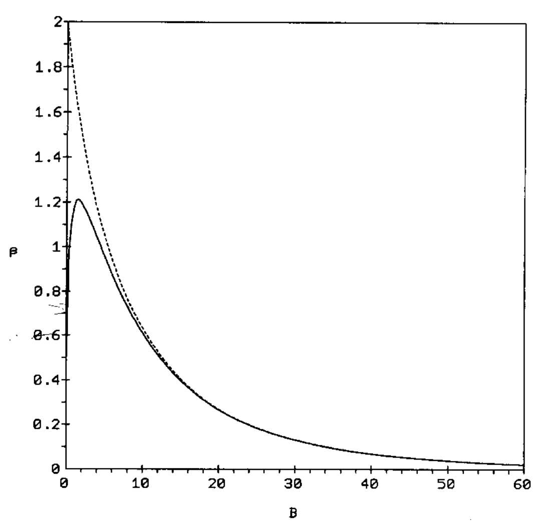

Fig. 2. β for B large. —— asymptotic solution; ------ numerical solution.

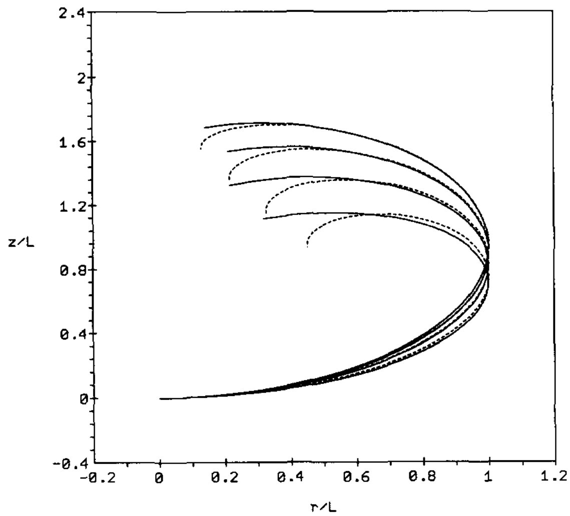

Fig. 3. Drop shapes for B small ($B = 0.25$, 0.5, 1, 2). —— asymptotic solution; ----- numerical solution.

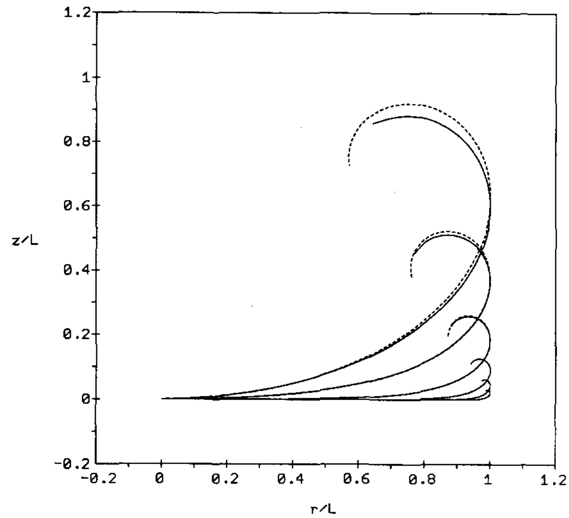

Fig. 4. Drop shapes for B large ($B = 4$, 16, 64, 256, 1024, 4096). —— asymptotic solution; ----- numerical solution.

6. Conclusions

The problem of the sessile drop is solved analytically by asymptotic series for small and large Bond number B, corresponding to large and small surface tension.

The problem for large B is of singular perturbation type, with a boundary layer of size $O(B^{-1/2})$ at the edge region. The middle area (i.e., the outer region) is exponentially flat, resulting in an outer solution which is either of perturbation series type with coefficients all zero, or the (non-zero, but exponentially small) solution of the linearized problem. The latter is clearly preferable, provided we match inner and outer solution logarithmically.

The problem for small B is a regular perturbation problem, with the approximation uniformly valid. This problem has been treated before by other authors, and our solution is, apart from some small improvements, not basically different. We have, however, included our solution for reference, and to complement the solution for large B.

By comparing with numerical solutions the accuracy of the approximations has been estimated.

Appendix. *Proof of*: $\psi_s > 0$ for $0 \leq \psi \leq \pi$.

By Taylor expansion near $s = 0$ we obtain: $\psi_s(0) = \frac{1}{2} p_0/\gamma$. Suppose that $p_0 < 0$. Then ψ and z will first decrease, but eventually has to become positive. So there is a point where $\psi = 0$ and $z < 0$. However, here $\psi_s = (p_0 + \rho g z)/\gamma < 0$, so ψ can only further decrease. Therefore, $p_0 > 0$.

Now the result follows from equation (4), rewritten into

$$2\gamma\psi_s = p_0 + \rho g z + \rho g r^{-2} \int_0^s r^2 \sin\psi \, ds' ,$$

which is evidently positive for $0 < \psi < \pi$ where $\sin\psi > 0$.

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