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1-D REFLECTION AT AN IMPEDANCE WALL

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The 1-D reflection of an arbitrary incident acoustic wave at an impedance wall is studied, resulting in the formulation of necessary conditions for physically possible impedance models. These results are applied to the problem of an impedance of Helmholtz resonator type, in which case the reflection of an incident pulse is obtained in explicit form, expressed in generalized Laguerre polynomials.

1. INTRODUCTION

For theoretical investigations of a harmonic (i.e., of a single frequency) sound wave reflecting at an impedance wall, the only information about the impedance which is required is its complex value Z [1]. However, for a general incident wave, with an extended frequency spectrum, one needs to know $Z = Z(\omega)$ as a function of frequency ω. Also in the context of a mean flow with inherent instabilities of the vortex sheet along the impedance wall, knowledge of the behaviour of $Z(\omega)$ in the (complex) frequency domain is essential [2].

Not any model $Z(\omega)$ is physically possible, however. It may also occur that the same model allows several interpretations (in the sense that it is obtained via several limits) of which only one is physical.

In the present paper the general solution of an arbitrary one-dimensional sound wave reflecting at an impedance wall is discussed. Specifically, the conditions on $Z(\omega)$ necessary to obtain a real and causal sound field are considered. The results are applied to the problem of a pulse reflecting at an impedance wall of Helmholtz resonator type. An explicit solution is obtained, and numerically evaluated examples of it are presented.

2. GENERAL PROBLEM

Consider the one-dimensional acoustic wave equations (made dimensionless on mean sound speed, mean density and some length) for potential ϕ, pressure p, and velocity v,

$$\phi_{xx} - \phi_{tt} = 0, \qquad p = -\phi_t, \qquad v = \phi_x \tag{1}$$

on $-\infty < x \leqslant 0$, $-\infty < t < \infty$, with the impedance boundary condition at $x = 0$

$$\hat{p}(0, \omega) = Z(\omega)\hat{v}(0, \omega), \tag{2}$$

where "^" denotes a Fourier transform on time, $\hat{p}(x, \omega) = \int_{-\infty}^{\infty} p(x, t)\, e^{-i\omega t}\, dt$. The complex impedance Z represents the reflection properties of the wall at $x = 0$. If the wall is passive, it has $\operatorname{Re} Z \geqslant 0$, while it absorbs acoustic energy if $\operatorname{Re} Z > 0$ [1]. A hard wall corresponds to $|Z| = \infty$. Except for the case of Z a real constant, Z must vary (in some way) with ω. This will be further discussed in the next section.

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3. GENERAL SOLUTION AND CONDITIONS FOR Z

The classical solution [1] of the one-dimensional wave equation is a sum of arbitrary functions of $t-x$ and $t+x$. So here one has the reflected field $g(t+x)$ of an incident field $f(t-x)$ satisfying

$$p(x, t)=f(t-x)+g(t+x), \qquad v(x, t)=f(t-x)-g(t+x). \tag{3}$$

If one introduces (formally, for the moment)

$$z(t)=(1/2\pi)\int_{-\infty}^{\infty} Z(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega, \tag{4}$$

one can transform the impedance boundary condition back to the time domain, leading to the convolution

$$p(0, t)=\int_{-\infty}^{\infty} z(\tau) v(0, t-\tau)\, \mathrm{d}\tau, \tag{5}$$

and hence

$$f(t)+g(t)=\int_{-\infty}^{\infty} z(t-\tau)\{f(\tau)-g(\tau)\}\, \mathrm{d}\tau, \tag{6}$$

which is, of course, just an integral equation in g for given f. A solution is found by Fourier transformation of equation (3), and then solving equation (2), yielding

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{Z(\omega)-1}{Z(\omega)+1} \hat{f}(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{7}$$

This is rather formal, however. It will be seen in the next section that sometimes equation (6) is ideally suited to obtain, more directly, explicit results.

Apart from equation (6) there is another important result drawn from equation (5): not any Z is physically possible, and one can deduce necessary conditions for $Z(\omega)$: since p and v are real, so must be z, and therefore Z has to satisfy the *reality* condition

$$Z^{*}(\omega)=Z(-\omega), \tag{8}$$

where Z^{*} denotes Z's complex conjugate; since $p(0, t)$ can only depend on values of $v(0, t)$ of the past, z has to satisfy the *causality* condition

$$z(t)=0 \qquad \text{for} \qquad t<0. \tag{9}$$

These conditions may, for example, be used to choose consistent parametrizations of measured impedances. These conditions will be applied here to select a correct interpretation of a non-uniform limit.

4. IMPEDANCE OF HELMHOLTZ RESONATOR TYPE

An impedance wall of Helmholtz resonator type with a thin facing sheet appears to be well described by

$$Z(\omega)=R+\mathrm{i}\omega m-\mathrm{i} \operatorname{cotg}(\omega L) \tag{10}$$

for positive parameters R, m and L [3]. $R+\mathrm{i}\omega m$ is the impedance change across the porous facing sheet (in practice, determined experimentally), and L is the depth of the resonator cell. It is easily verified that Z satisfies the reality condition (8). It is, however, not so trivial whether or how it satisfies the causality condition (9), since the integral for

z is divergent. The integral does have a meaning, however, if one considers z as a generalized function [4]. Then a causal z is obtained if one interprets

$$-\mathrm{i}\,\mathrm{cotg}\,(\sigma) = 1 + 2 \lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} \exp\,(-2\mathrm{i}n\sigma - \varepsilon n) \tag{11}$$

with the result

$$z(t) = R\delta(t) + m\delta'(t) + \delta(t) + 2 \sum_{n=1}^{\infty} \delta(t_n) \tag{12}$$

where $t_n = t - 2nL$. For the general solution one obtains from equation (6) the equation

$$(R+2)g(t) + mg'(t) = Rf(t) + mf'(t) + 2 \sum_{n=1}^{\infty} \{f(t_n) - g(t_n)\} \tag{13}$$

with solution ($m \neq 0$)

$$g(t) = f(t) - \frac{2}{m} \int_{-\infty}^{t} \exp\left(-\frac{R+2}{m}(t-\tau)\right)\left[f(\tau) - \sum_{n=1}^{\infty} \{f(\tau_n) - g(\tau_n)\}\right] \mathrm{d}\tau \tag{14a}$$

and ($m = 0$)

$$g(t) = \frac{R}{R+2} f(t) + \frac{2}{R+2} \sum_{n=1}^{\infty} \{f(t_n) - g(t_n)\}. \tag{14b}$$

So the response $g(t)$ is expressed in $f(t)$ and in f and g of the past. If $f \equiv 0$ (and hence $g \equiv 0$) before some time, the series in expressions (14a) and (14b) are finite and the solution g can be built up iteratively, in time intervals of $2L$. This can be made explicit for a pulse, and after some (tedious) algebra one obtains (see the Appendix) if $m \neq 0$

$$f(t) = \delta(t), \qquad g(t) = \delta(t) - \frac{2}{m} \sum_{n=0}^{[t/2L]} \exp\left(-\frac{R+2}{m} t_n\right) \mathrm{L}_n^{(-1)}(2t_n/m) \tag{15}$$

with [ ] denoting the integral part, and where $\mathrm{L}_n^{(-1)}$ is a suitably defined generalized Laguerre polynomial [5]:

$$\mathrm{L}_n^{(-1)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n-1}{k-1} x^k \tag{16}$$

with $\binom{n-1}{-1} = 0$ if $n \geqslant 1$, and $= 1$ if $n = 0$. Use is made of the relation

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{L}_n^{(-1)}(x) = -\sum_{k=0}^{n-1} \mathrm{L}_k^{(-1)}(x). \tag{17}$$

Convenient for numerical evaluation is the recurrence relation

$$(n+1)\mathrm{L}_{n+1}^{(-1)} = (2n - x)\mathrm{L}_n^{(-1)} - (n-1)\mathrm{L}_{n-1}^{(-1)} \tag{18}$$

starting with $\mathrm{L}_0^{(-1)} = 1$, $\mathrm{L}_1^{(-1)} = -x$.

If $m = 0$ one has the expression

$$g(t) = \frac{R}{R+2} \delta(t) + \frac{4}{R(R+2)} \sum_{n=1}^{\infty} \left(\frac{R}{R+2}\right)^n \delta(t_n) \tag{19}$$

consisting of a row of δ-pulses. This can be derived directly from equation (14b), or (for example as a check) from equation (15) in the limit $m \to 0$, by using the relation

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \mathrm{H}(x) \exp(-\lambda x/\varepsilon) \mathrm{L}_n^{(-1)}(x/\varepsilon) = \begin{Bmatrix} \dfrac{1}{\lambda}\,\delta(x) & \text{if } n = 0 \\ -\dfrac{(\lambda-1)^{n-1}}{\lambda^{n+1}}\,\delta(x) & \text{if } n \geq 1 \end{Bmatrix}, \tag{20}$$

where $\lambda > 0$.

The addition of more and more pulses in the region $x < 0$, which is due to the multiple reflections inside the resonator cell, suggests an infinitely long echo of a single pulse (see equations (15), (19)). But of course, the energy content decreases, and physically there will be no detectable signal left for $t \to \infty$.

Although there is, for any $0 < R < \infty$, acoustic energy dissipated by the impedance wall [1], it is interesting to note that for the present impedance there is some conservation

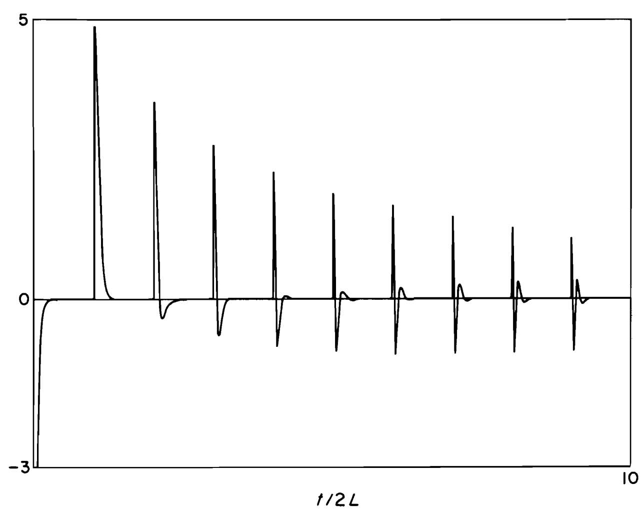

Figure 1. $g(t)$ with $R = 1$, $m/2L = 0{\cdot}1$.

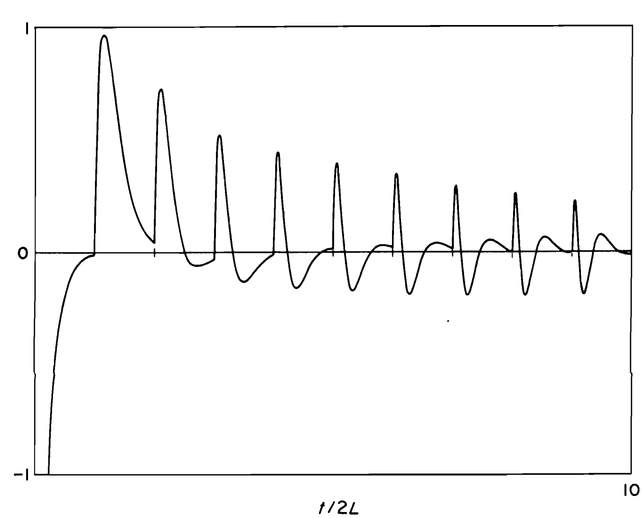

Figure 2. $g(t)$ with $R = 1$, $m/2L = 0{\cdot}5$.

law valid, in the form of equal average values of incident and reflected waves:

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = \hat{f}(0) = \lim_{\omega\to 0} \frac{Z(\omega)-1}{Z(\omega)+1}\hat{f}(0) = \hat{g}(0) = \int_{-\infty}^{\infty} g(t)\,\mathrm{d}t. \tag{21}$$

Of course, one might also say that the wall is hard ($|Z|=\infty$) if $\omega=0$, and therefore reflects the zero-frequency component completely.

5. EXAMPLES

Some numerically evaluated examples of equation (15) are given in Figures 1-8 and of equation (19) (the amplitudes) in Figures 9-11. In all cases the time scale is $t/2L$, and the parameters varied are R and $m/2L$. (This is possible because there is in the problem no length scale other than L.)

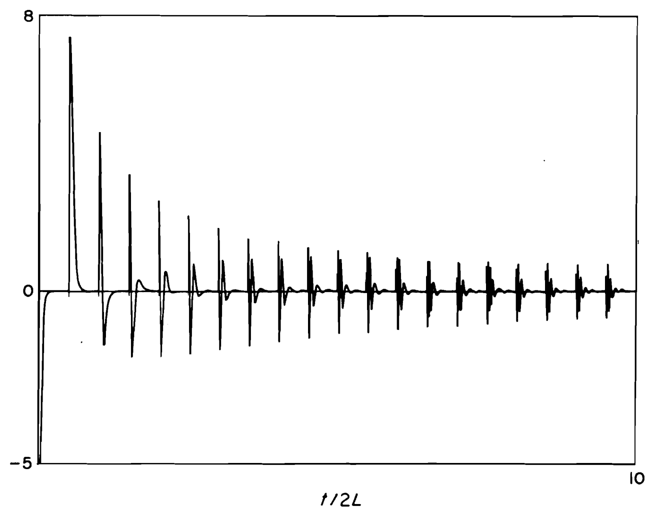

Figure 3. $g(t)$ with $R=0$, $m/2L=0{\cdot}1$.

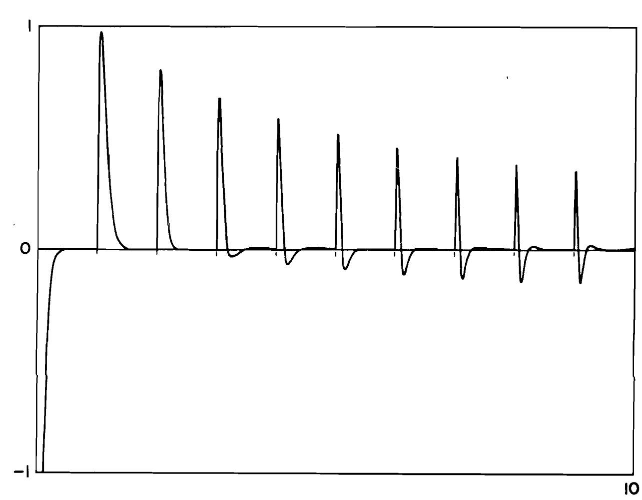

Figure 4. $g(t)$ with $R=3$, $m/2L=0{\cdot}3$.

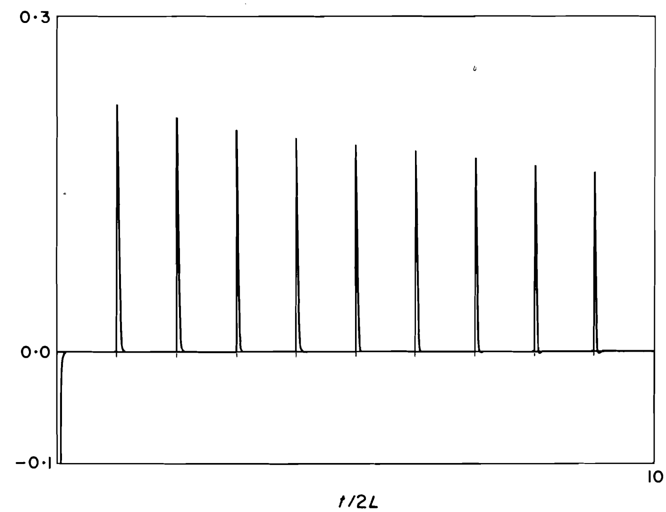

Figure 5. $g(t)$ with $R=20$, $m/2L=0{\cdot}3$.

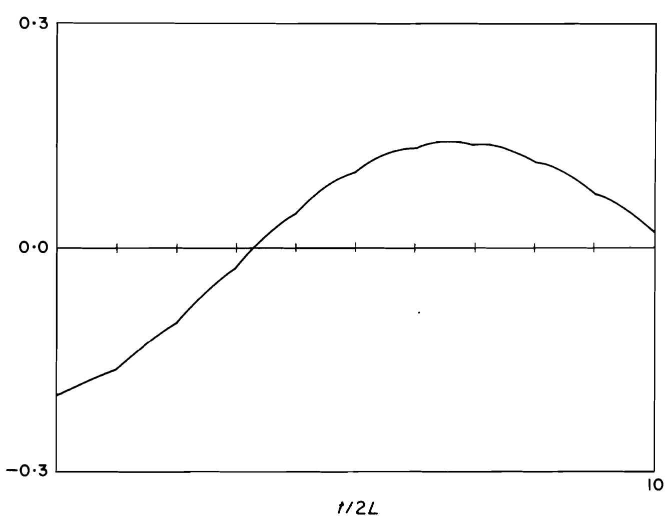

Figure 6. $g(t)$ with $R=0$, $m/2L=10$.

One sees that R and $m/2L$ are in some sense counter-effective: an increasing R as well as a decreasing $m/2L$ enhance the character of a train of isolated pulses (Figures 1, 2, 5 and 7), while a decreasing R or increasing $m/2L$ smooths the reflected signal (Figure 6). When R and $m/2L$ are small enough, the response takes the form of a train of a finite number of damped oscillations (Figures 1, 2, 3, 4 and 8).

For $m=0$ (equation (19)) it is not difficult to prove that always the first two pulses are higher in amplitude than the others, but which of the two is highest depends on R. If $R>\sqrt{5}-1=1{\cdot}236$ (Figure 9) the first pulse is the highest; if $R=\sqrt{5}-1$ (Figure 10) the first and second are equal, and if $0<R<\sqrt{5}-1$ the second pulse is the highest. If $R\to\infty$,

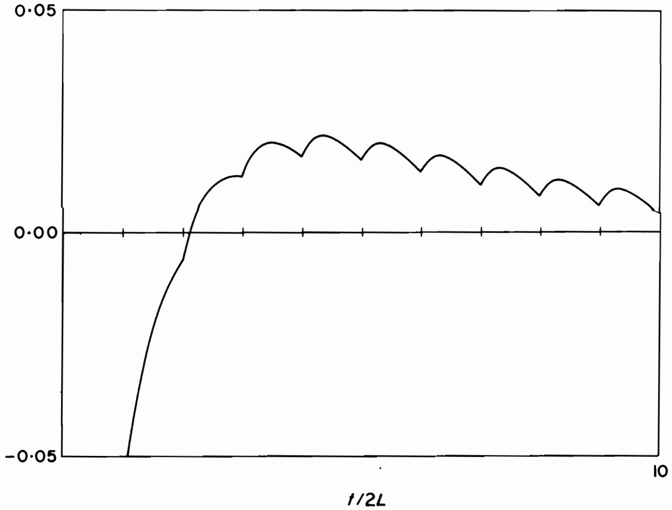

Figure 7. $g(t)$ with $R = 10$, $m/2L = 10$.

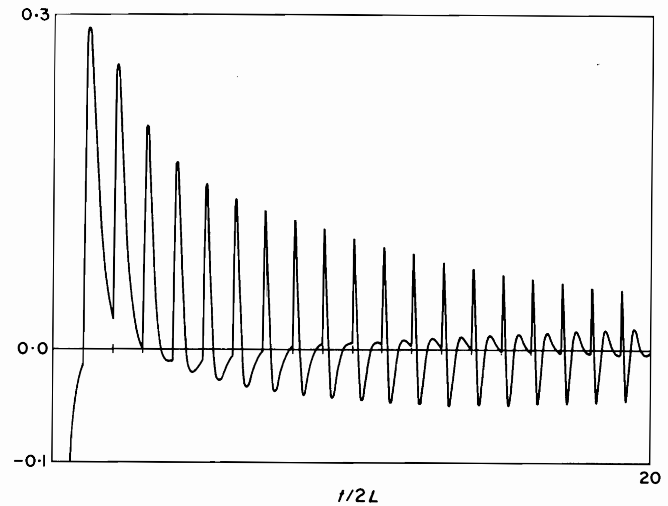

Figure 8. $g(t)$ with $R = 3$, $m/2L = 1$.

only the first pulse is present (hard wall reflection), and with $R = 0$ there is only the second pulse (reflection at the hard walled bottom of the resonator cell).

6. CONCLUSIONS

By studying the reflection of a one-dimensional acoustic wave at an impedance wall, necessary conditions have been derived for the impedance as a function of frequency. These conditions, indicating which impedance models are *a priori* physically impossible, have been applied to the problem of an impedance of Helmholtz resonator type. In this

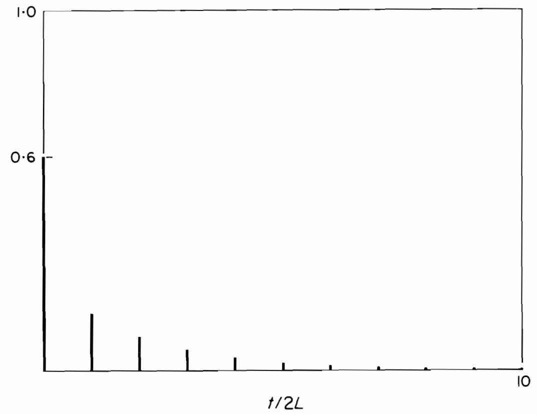

Figure 9. Amplitudes of $g(t)$, with $R=3$, $m/2L=0$.

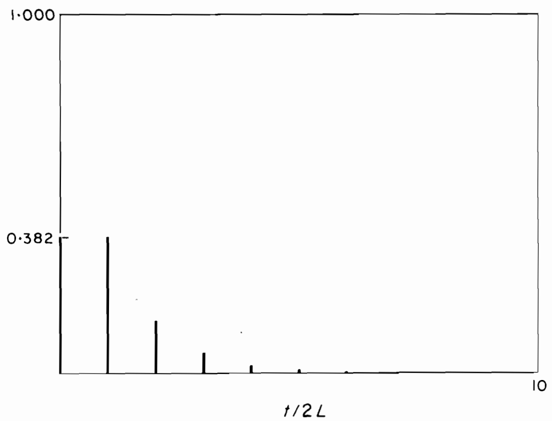

Figure 10. Amplitudes of $g(t)$, with $R=1{\cdot}236$, $m/2L=0$.

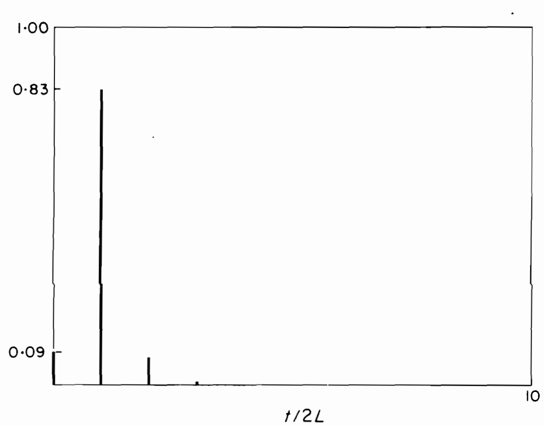

Figure 11. Amplitudes of $g(t)$, with $R=0{\cdot}2$, $m/2L=0$.

case the impedance model used has to be interpreted as a certain limit, selected by the above conditions. Then, the explicit solution could be derived for the reflection of a pulse, in terms of a finite series of generalized Laguerre polynomials. This solution was illustrated by some examples.

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APPENDIX

Some details of the derivation of solution (15) of equation (14a) are as follows. First one defines

$$h(t)=\tfrac{1}{2}m\exp\left(t(R+2)/m\right)(f(t)-g(t)),$$

$$a=\exp\left(2L(R+2)/m\right),\qquad b=2/m,\qquad N=[t/2L].$$

Then $h=0$ for $t<0$, while for $t\geqslant 0$

$$h(t)=1-b\sum_{n=1}^{N}a^n\int_0^{t_n}h(\tau)\,\mathrm{d}\tau.$$

This suggests that h takes the form

$$h(t)=\sum_{n=0}^{N}a^n h_n(bt_n)$$

for some h_n. Then one has

$$h(t)=1-b\sum_{n=1}^{N}a^n\left[\sum_{m=0}^{N-n-1}\sum_{k=0}^{m}\int_{2mL}^{2(m+1)L}a^k h_k(b\tau_k)\,\mathrm{d}\tau+\sum_{k=0}^{N-n}\int_{2(N-n)L}^{t_n}a^k h_k(b\tau_k)\,\mathrm{d}\tau\right].$$

Interchange k- and m-summation, and change to a new variable of integration, so that

$$h(t)=1-\sum_{n=1}^{N}\sum_{k=0}^{N-n}a^{n+k}\int_0^{bt_{n+k}}h_k(\tau)\,\mathrm{d}\tau=1-\sum_{n=1}^{N}\sum_{k=n}^{N}a^k\int_0^{bt_k}h_{k-n}(\tau)\,\mathrm{d}\tau.$$

Again interchanging n- and k-summation yields

$$h(t)=1-\sum_{n=1}^{N}a^n\int_0^{bt_n}\sum_{k=0}^{n-1}h_k(\tau)\,\mathrm{d}\tau.$$

Finally, with the use of relation (17) it is clear that this equation has a solution $h_n=\mathrm{L}_n^{(-1)}$, while uniqueness is guaranteed by the iterative way $h(t)$ is built up from the past.