659. Show that

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+\lambda^2)(n+a^2\lambda^2)} = \frac{\pi}{1+a}\lambda^{-1} - \frac{\zeta(3/2)}{4\pi a^2}\lambda^{-4} + O(\lambda^{-6}), \quad (\lambda\to\infty)$$

where $a > 0$.

(S.W. RIENSTRA)

Solutions by J. BOERSMA (2x), J.A. VAN CASTEREN, A.A. JAGERS (2x), J.B.M. MELISSEN, N. ORTNER, S.W. RIENSTRA, N.M. TEMME.

Most solutions produce a complete asymptotic series. J.A. VAN CASTEREN presents a general and rather complicated result using contour integration for its proof. S.W. RIENSTRA applies the Poisson summation formula to $f(x) = (x+1)^{-1}(x+a^2)^{-1} x^{\frac{1}{2}} H(x)$, where H denotes the Heaviside step function. N. ORTNER studies the sum $\Sigma_{n=1}^{\infty} n^{\alpha}/[(n+a_1)\ldots(n+a_m)]$, with all a_i's > 0, $a_i \neq a_j$ $(i\neq j)$, $m \geq 2$, $\alpha < m-1$, by means of the Mellin transform.

This tool is also used by J. BOERSMA (1x) and J.B.M. MELISSEN. In his other solution J. BOERSMA uses properties of Lerch's transcendent Φ. A.A. JAGERS applies the Euler-Maclaurin sum formula. N.M. TEMME studies the integral $(2i)^{-1}\int_C \cot\pi z\, f(z)dz$ where $f(z) := (z+\lambda^2)^{-1}(z+a^2\lambda^2)^{-1} z^{\frac{1}{2}}$ and C encircles the positive integers.

SOLUTION by J. BOERSMA.

Using the Mellin transform

$$M\left\{\frac{x^{1/2}}{x+a}\right\} = \int_0^\infty \frac{x^{s-1/2}}{x+a}\,dx = \frac{\pi a^{s-1/2}}{\cos(\pi s)}, \; -\tfrac{1}{2} < \operatorname{Re} s < \tfrac{1}{2}$$

we have the following representation of the summand by an inverse Mellin transform

$$\frac{x^{1/2}}{(x+\lambda^2)(x+a^2\lambda^2)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\cos(\pi s)}\,\frac{1-a^{2s-1}}{a^2-1}\,\lambda^{2s-3}x^{-s}\,ds$$

where $-\frac{1}{2} < c < \frac{3}{2}$. Let the sum of the series be denoted by $S(\lambda)$, then because of $\Sigma_{n=1}^{\infty} n^{-s} = \zeta(s)$ for $\operatorname{Re} s > 1$, we are led to the integral representation

$$S(\lambda) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi\zeta(s)}{\cos(\pi s)}\,\frac{1-a^{2s-1}}{a^2-1}\,\lambda^{2s-3}\,ds, \quad 1 < c < 3/2.$$

In the latter integral the path of integration is shifted to the left, thus intercepting simple poles at $s = 1$, due to $\zeta(s)$, and at $s = \frac{3}{2} - m$, $m = 2,3,4,\ldots$, due to $\cos(\pi s)$. The corresponding residue series provides the asymptotic expansion of $S(\lambda)$ as $\lambda \to \infty$, viz.

$$S(\lambda) \sim \frac{\pi}{1+a}\lambda^{-1} + \sum_{m=2}^{\infty} (-1)^m\,\zeta(\tfrac{3}{2}-m)\,\frac{1-a^{2-2m}}{a^2-1}\,\lambda^{-2m}, \quad \lambda \to \infty.$$

Here the second term of the asymptotic series can be expressed as

$$\zeta(-\tfrac{1}{2})\,a^{-2}\lambda^{-4} = -\frac{\zeta(3/2)}{4\pi a^2}\lambda^{-4}$$

by means of Riemann's functional equation for the zeta-function.

REMARK. By shifting the path of integration to the right, one obtains the expansion

$$S(\lambda) = \sum_{m=0}^{\infty} (-1)^m\,\zeta(m+\tfrac{3}{2})\,\frac{a^{2m+2}-1}{a^2-1}\,\lambda^{2m}$$

which can be shown to be convergent for $0 \le \lambda^2 < \min(1, a^{-2})$.