877. A well-known corollary of Murphy's Law is the fact that for a bicyclist, wind is more often a disadvantage than an advantage. In order to verify this empirical law theoretically, we investigate the following model:

An object moves in $\mathbb{R}^2$ with constant velocity $-\vec{v} = (-V, 0)$ where $V > 0$ in a uniform wind field with velocity $\vec{w} = (W\cos\theta,\ W\sin\theta)$, where $W \geq$

0, $0 \leq \theta \leq \pi$. The total effective velocity is $\vec{u} := \vec{v} + \vec{w}$. We assume the total air resistance $\vec{F}$ to be equal to $c|\vec{u}|\vec{u}$, where $c$ is a positive constant. The drag adverse to the bicyclist's motion is the component of $\bar{F}$ in the $x$-direction, say $D(\theta, W, V)$. We introduce the quantity $\mu := W/V$.

1. Determine the angle $\theta =: \theta_1(\mu)$ for which
$$D(\theta, W, V) = D(\theta, 0, V).$$
2. Determine the value of $\mu$ for which $\theta_1(\mu)$ attains its maximum.

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By using the quantity $\mu = W/V$, the equation $D(\theta, W, V) = D(\theta, 0, V)$ can be written in the form
$$\sqrt{(1 + \mu\cos\theta)^2 + (\mu\sin\theta)^2}(1 + \mu\cos\theta) = 1$$
In order to solve the equation for $\theta$ as a function $\theta_1(\mu)$ of $\mu$ and to find the value $\mu$ for which $\theta_1(\mu)$ is maximal, we first substitute $x = 1 + \mu\cos\theta$, $y = \mu\sin\theta$ in this equation. Then, see the figure below, the solution $\theta_1(\mu)$ corresponds to the intersection point $P$ of the circle $C : (x-1)^2 + y^2 = \mu^2$ and the curve $K : f(x, y) := x\sqrt{x^2 + y^2} = 1$. Furthermore, the value $\mu$ for which $\theta_1(\mu)$ is maximal corresponds to the situation where the line $MP$ is tangent to the curve $K$ at the point $P$.

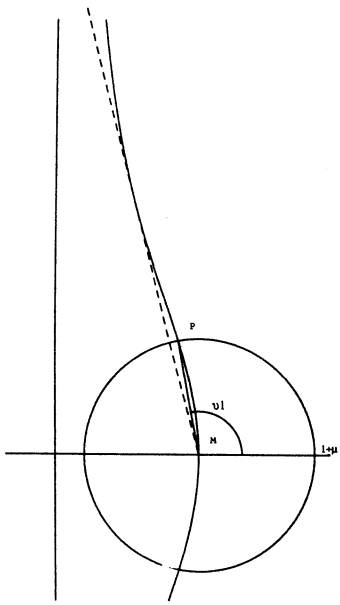


FIGURE 1.

It follows that the $x$-coordinate $x_1(\mu)$ of $P$ must satisfy the equation

$$2x^3 + (\mu^2 - 1)x^2 - 1 = 0 \qquad (0 < x \leq 1)$$

One may solve this equation on several lucky ways or by using Cardano's formula.
The wanted solution $x_1(\mu)$ is given by

$$x_1(\mu) = \left( \sqrt[3]{1 + \sqrt{1 - \alpha}} + \sqrt[3]{1 - \sqrt{1 - \alpha}} \right)^{-1}$$

$$\alpha = (\frac{\mu^2 - 1}{3})^3$$

In case $\alpha > 1$ (i.e. $\mu > 2$) one may take the principal values of the roots in question.
Consequently, one has

$$\theta_1(\mu) = \arccos(\frac{x_1(\mu) - 1}{\mu})$$

Now, we return to the problem of finding the value for which $\theta_1(\mu)$ is maximal. This implies that grad $f$ must be perpendicular to $MP$ at the point $P = (x_1, y_1)$ which leads to the two equations

$$x_1\sqrt{x_1^2 + y_1^2} = 1$$

$$(2x_1^2 + y_1^2)(x_1 - 1) + x_1 y_1^2 = 0$$

By eliminating $y_1$ we get

$$x_1^4 - 2x_1 + 1 = (x_1 - 1)(x_1^3 + x_1^2 + x_1 - 1) = 0$$

The solution $x_1 \in (0, 1)$ is given by $x_1 = 0.543689....$ .
Hence $\mu = \sqrt{(x_1 - 1)^2 + y_1^2} = \sqrt{1 - 2x_1 + 1/x_1^2} = 1.81538...$, and, finally

$$\theta_1 = 1.82448... \qquad (\approx 104.6^\circ)$$