678. Show that

a) $$\int_0^\infty \left[1 - \frac{2}{\pi x}\frac{1}{J_\nu^2(x)+Y_\nu^2(x)}\right] dx = (\tfrac{1}{2}\nu-\tfrac{1}{4})\pi,$$

b) $$\int_0^\infty \left[1 - \frac{2}{\pi x}\frac{1-\nu^2/x^2}{\{J'_\nu(x)\}^2+\{Y'_\nu(x)\}^2}\right] dx = (\tfrac{1}{2}\nu+\tfrac{1}{4})\pi,$$

where J_ν and Y_ν are Bessel functions of the first and second kind, respectively, of order $\nu \ge 0$.

(S.W. RIENSTRA)

Solutions by J. BOERSMA, P.J. DE DOELDER, A.A. JAGERS, S.W. RIENSTRA.

The solutions by P.J. DE DOELDER, A.A. JAGERS are more or less similar to the one by J. BOERSMA.

SOLUTION by J. BOERSMA.

By means of the Wronskian-type relations ([1], form. 3.63 (1),(6))

$$J_\nu(x)\,Y'_\nu(x) - Y_\nu(x)\,J'_\nu(x) = \frac{2}{\pi x},$$

$$J'_\nu(x)\,Y''_\nu(x) - Y'_\nu(x)\,J''_\nu(x) = \frac{2}{\pi x}\left(1-\frac{\nu^2}{x^2}\right),$$

it is readily seen that

$$\frac{d}{dx}\left[\arctan\frac{Y_\nu(x)}{J_\nu(x)}\right] = \frac{2}{\pi x}\frac{1}{J_\nu^2(x)+Y_\nu^2(x)},$$

$$\frac{d}{dx}\left[\arctan\frac{Y'_\nu(x)}{J'_\nu(x)}\right] = \frac{2}{\pi x}\frac{1-\nu^2/x^2}{\{J'_\nu(x)\}^2+\{Y'_\nu(x)\}^2}.$$

The first arctan-function is increasing for $x \ge 0$; its value at $x = 0$ is $-\pi/2$ and the function jumps from $+\pi/2$ to $-\pi/2$ at the successive positive zeros $j_{\nu,n}$ $n = 1, 2, 3,\ldots,$ of $J_\nu(x)$. Likewise, arctan $(Y'_\nu(x)/J'_\nu(x))$ is decreasing for $0 \le x \le \nu$ and increasing for $x \ge \nu$, starting with the value $+\pi/2$ at $x = 0$ if $\nu > 0$. The second arctan-function jumps from $+\pi/2$ to $-\pi/2$ at the successive positive zeros $j'_{\nu,n}, n = 1,2,3,\ldots,$ of $J'_\nu(x)$; here, it is used that $j'_{\nu,1} > \nu$, ([1], form. 15.3(1)).

By means of these properties of the arctan-functions it is easily found that

$$\int_0^{j_{\nu,n}} \left[1 - \frac{2}{\pi x}\,\frac{1}{J_\nu^2(x) + Y_\nu^2(x)}\right]dx = j_{\nu,n} - n\pi,$$

$$\int_0^{j'_{\nu,n}} \left[1 - \frac{2}{\pi x}\,\frac{1-\nu^2/x^2}{\{J'_\nu(x)\}^2 + \{Y'_\nu(x)\}^2}\right]dx = j'_{\nu,n} - (n-1)\pi, \quad \nu > 0.$$

In the latter integrals let $n \to \infty$, then by means of McMahon's expansions ([2], form. 9.5.12, 9.5.13)

$$j_{\nu,n} = (n + \tfrac{1}{2}\nu - \tfrac{1}{4})\pi + O(\tfrac{1}{n}), \quad j'_{\nu,n} = (n+\tfrac{1}{2}\nu - \tfrac{3}{4})\pi + O(\tfrac{1}{n}), \quad (n\to\infty)$$

the results to be shown are obvious. The second result with $\nu = 0$ follows from the first result with $\nu = 1$, since $J'_0(x) = -J_1(x)$, $Y'_0(x) = -Y_1(x)$.

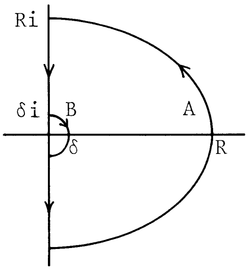
[1] WATSON, G.N., *A treatise on the Theory of Bessel Functions*, Cambridge, 1958.

[2] ABRAMOWITZ, M. & I.A. STEGUN, *Handbook of Mathematical Functions*, Washington, 1965.

SOLUTION. (Based on S.W. RIENSTRA's solution).
The modified Bessel function of the third kind $K_\nu(z)$, $\nu \geq 0$, has ([1], p. 62) no zeros for which $|\arg z| \leq \frac{1}{2}\pi$. Therefore we have

$$\frac{1}{2i}\int_C \left(1 + \frac{K'_\nu(z)}{K_\nu(z)}\right)dz = 0,$$

where C is the contour shown in the figure.



Using ([2], form. 9.7.2, 9.7.4, 9.6.9, 9.6.8)

(1) $\dfrac{K'_\nu(z)}{K_\nu(z)} = -1 - \tfrac{1}{2}z^{-1} + O(z^{-2})$, $z \to \infty$, $|\arg z| < \tfrac{3}{2}\pi$,

(2) $\dfrac{K'_\nu(z)}{K_\nu(z)} \sim -\nu z^{-1}$, $z \to 0$, $\nu > 0$, and

(3) $\dfrac{K'_0(z)}{K_0(z)} \sim (z \log z)^{-1}$, $z \to 0$,

we obtain

$$\lim_{R\to\infty} \frac{1}{2i} \int_A \left(1 + \frac{K'_\nu(z)}{K_\nu(z)} \right) dz = -\tfrac{1}{4}\pi ,$$

$$\lim_{\delta\downarrow 0} \frac{1}{2i} \int_B \left(1 + \frac{K'_\nu(z)}{K_\nu(z)}\right) dz = \tfrac{1}{2}\nu\pi , \quad \nu \geq 0.$$

The contribution to $\frac{1}{2i} \int_C \left(1 + \frac{K'_\nu(z)}{K_\nu(z)} \right) dz$ of the two parts of C along the imaginary axis equals

$$- \int_\delta^R \left(1 + \frac{K'_\nu(iy)K_\nu(-iy) + K'_\nu(-iy)K_\nu(iy)}{2\, K_\nu(iy)K_\nu(-iy)} \right) dy,$$

where ([1], §7.2, form. (15), (16); §7.11, form. (30))

$$K_\nu(iy)K_\nu(-iy) = \tfrac{1}{4}\pi^2 (J_\nu^2(y) + Y_\nu^2(y)) ,$$

$$K'_\nu(iy)K_\nu(-iy) + K'_\nu(-iy)K_\nu(iy) = - \pi y^{-1} .$$

With these results it follows that

$$\int_0^\infty \left[1 - \frac{2}{\pi y} \, \frac{1}{J_\nu^2(y) + Y_\nu^2(y)} \right] dy = (\tfrac{1}{2}\nu - \tfrac{1}{4})\pi , \quad \nu \geq 0.$$

To derive formula b) we use the fact that the function $K'_\nu(z)$, $\nu \geq 0$, has no zeros in the right-half of the complex plane ([1], pp. 62-63).
Hence we have

$$\frac{1}{2i} \int_C \left(1 + \frac{K''_\nu(z)}{K'_\nu(z)} \right) dz = 0.$$

Writing

$$1 + \frac{K''_\nu(z)}{K'_\nu(z)} = 1 - \frac{1}{z} + \left(1 + \frac{\nu^2}{z^2} \right) \frac{K_\nu(z)}{K'_\nu(z)} ,$$

it follows from (1), (2) and (3) that

$$\lim_{R\to\infty} \frac{1}{2i} \int_A \left(1 + \frac{K''_\nu(z)}{K'_\nu(z)} \right) dz = - \tfrac{1}{4}\pi,$$

$$\lim_{\delta\downarrow 0} \frac{1}{2i} \int_B \left(1 + \frac{K''_\nu(z)}{K'_\nu(z)} \right) dz = \tfrac{1}{2}\nu\pi + \tfrac{1}{2}\pi , \quad \nu \geq 0.$$

Furthermore,

$$\frac{1}{2i}\left\{\int_{Ri}^{\delta i} + \int_{-\delta i}^{-Ri}\right\}\left(1+\frac{K''_\nu(z)}{K'_\nu(z)}\right) dz \text{ is equal to}$$

$$-\int_{\delta}^{R}\left[1+\tfrac{1}{2}\left(1-\frac{\nu^2}{y^2}\right)\frac{K_\nu(iy)K'_\nu(-iy)+K_\nu(-iy)K'_\nu(iy)}{K'_\nu(iy)K'_\nu(-iy)}\right] dy,$$

where ([1], §7.2, form. (15), (16))

$$K'_\nu(iy)\,K'_\nu(-iy) = \tfrac{1}{4}\pi^2\left[(J'_\nu(y))^2 + (Y'_\nu(y))^2\right].$$

The result to be proved now follows in an obvious manner.

[1] ERDÉLYI, A., et al., *Higher Transcendental Functions*, vol. II, New York, 1953.

[2] ABRAMOWITZ, M. & I.A. STEGUN, *Handbook of Mathematical Functions*, Washington, 1965.