738. Prove that for integers $k \geqslant 2$ there exist polynomials P_k such that

$$\sum_{n=1}^{N} n^k = N(N+1)(2N+1)P_k((2N+1)^2)$$

if k is even, and

$$\sum_{n=1}^{N} n^k = N^2(N+1)^2 P_k((2N+1)^2)$$

if k is odd.

(S. RIENSTRA)

Solutions by A.A. JAGERS, R.A. KORTRAM, S. RIENSTRA, H. ZANTEMA.

All solutions depend on certain properties of Bernoulli polynomials. The definition and elementary properties of Bernoulli polynomials can be found in E.T. WHITTAKER & G.N. WATSON, Modern Analysis, p. 126.

SOLUTION by A.A. JAGERS.
Let $C_m(x) := B_m((x+1)/2) - B_m(0)$, where $B_m(x)$ is the Bernoulli polynomial of degree m. Then

$$(k+1)\sum_{n=1}^{N} n^k = B_{k+1}(N+1) - B_{k+1}(0) = C_{k+1}(2N+1)$$

and so it suffices to prove that $C_{k+1}(x)/(x^2-1)$ is an odd polynomial in x if k is even and $\geqslant 2$, *and* that $C_{k+1}(x)/(x^2-1)^2$ is an even polynomial in x if k is odd and $\geqslant 3$. Now first of all $C_m(x)$ is a polynomial itself and for $m \neq 1$ we have $C_m(-x) = (-1)^m C_m(x)$ (the required symmetry property), because $B_m(1-t) = (-1)^m B_m(t)$ and $B_m(0) = 0$ for m odd , $m > 1$. Furthermore $x^2-1 = (x-1)(x+1)$ is a factor of $C_m(x)$ for all $m > 1$, since $C_m(-1) = 0$ by definition and $C_m(1) = B_m(1) - B_m(0) = 0$ for $m > 1$. Finally x^2-1 is even a double factor of $C_{k+1}(x)$ if k is odd, $k \geqslant 3$, because $C'_{k+1}(x) = \frac{1}{2}B'_{k+1}((x+1)/2) = \frac{1}{2}(k+1)B_k((x+1)/2) = \frac{1}{2}(k+1)C_k(x)$ in this case.