

## INTERMODAL RESONANCE OF IN-PLANE FREE VIBRATIONS OF SUSPENDED CABLES

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### ABSTRACT

The weakly nonlinear in-plane free vibrations of a single suspended cable, or a coupled system of suspended cables, may be classified as gravity modes (no tension variations to leading order) and elasto-gravity modes (tension and vertical displacement equally important). The gravity mode (probably the most common type of vibration of relatively inelastic spans) does not exist for particular values of the problem parameter. The reason is that for these parameter values the 1st and 2nd harmonic are in resonance.

This resonance has been analysed here in detail by an application of the Lindstedt-Poincaré technique. The leading order 1st and 2nd harmonic can only exist together with each other. As a result, the tension, albeit of 2nd harmonic, does not vanish at leading order and the mode is not anymore truly dominated by gravity alone.

The analysis is worked out here in detail for a single span but the results for a system of coupled spans seem to be similar. It is conjectured that designing the suspended cable with parameter values right at this resonance will delay or hinder the occurrence of galloping.

### 1. INTRODUCTION

Overhead transmission lines for transport of high-voltage electricity are cables made of aluminium alloy, suspended between high towers in the countryside. The part of a line that is suspended between two towers is called a span. In wintertime, when the cable is covered by snow or ice, the cables are vulnerable to large scale vertical vibrations in combination with a torsional vibration sustained by steady cross winds. This aero-elastic instability is known as galloping [1–16]. For high enough amplitudes neighbouring conductors may get close enough for the air-insulation to break down, causing a short-circuit and structural damage to the cables.

It is known from observations that even a small wind force is sufficient to maintain a galloping vibration. The motion of the cable is therefore very close to a free vibration, which is what we will consider here, although a more complete modelling should include the driving force of the wind and the effect of friction with the air [5, 16].

Both the torsional motion and the horizontal cable deflection are known to be important for the full problem, but we will concentrate on an asymptotic analysis of the vertical (*i.e.* in-plane) motion. Therefore, torsion will be assumed to be decoupled from the vertical vibration and the horizontal motion to be negligible.

The paper is organised as follows. First, we rederive the pertaining equations [3, 4] by asymptotic reduction of a full model of a single suspended cable. This has been presented earlier in [11, 12], but for clarity of notation and assumptions we repeat the arguments here.

The second part consists of a Lindstedt-Poincaré approximation of weakly nonlinear transversal in-plane wave motion, where the small parameter is now the dimensionless amplitude. We study the motion that is to leading order tension-free. A result, we reported already in [12], is the presence of intermodal resonance for certain choices of parameters.

Then we analyse this resonance. It is shown that near this resonance no tension-free motion is possible, since the second harmonic, which has non-zero tension, is of equal order of magnitude.

Finally a practical example case has been evaluated and illustrated by figures.

## 2. MODEL

### 2.1 Differential equations and boundary conditions

Consider a cable of unstretched length  $L$ , suspended between two fixed supports (see Figure 1), separated by a distance  $S$ , the span size. The cable is linearly elastic, with negligible bending stiffness, of uniform undeformed effective cross-section  $\mathcal{A}$ , mass per unit length  $m$ , and Young's modulus  $E$ . It has a length per span  $L$  when the cable is free of tension. Stationary, each span has a sag  $D$  which depends on  $S$  and  $L$ . In practice, the prescribed sag  $D$  is obtained by applying a suitable cable tension, from which length  $L$  follows. So  $L$  is an unknown of the stationary problem. This, however, has no bearing at all on the unsteady problem.

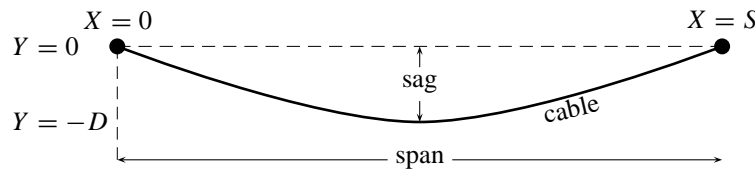


Figure 1: Sketch of a suspended cable.

We parametrize the position along the length of the cable by the variable  $\ell \in [0, L]$ , such that this is just the arc length when the cable is unstretched. The (dimensional) time variable is  $t$ . We will here only consider cable motion in a vertical plane, which is provided with a Cartesian coordinate system oriented such that the gravity vector  $-g\mathbf{e}_y$  points into the negative  $y$ -direction.

The cable position is given by the position vector  $\mathbf{X}(\ell, t) = (X(\ell, t), Y(\ell, t))$  with a corresponding tension vector  $\mathbf{T}(\ell, t) = T(\ell, t)(\cos \psi, \sin \psi)$ , where  $\psi$  is the positively oriented angle between the cable tangent and the horizontal. The tension vector is tangent to the cable because of the assumed negligible bending stiffness.

According to Hooke's law [17], a cable element  $d\ell$  is elongated in proportion to the tension (Figure 2), so

$$(dX^2 + dY^2)^{1/2} = \left(1 + \frac{T}{EA}\right)d\ell. \quad (1)$$

According to Newton's law, the internal (tension) and external (gravity) forces are in equilibrium with the inertial forces, so

$$d\mathbf{T} = (g\mathbf{e}_y + \ddot{\mathbf{X}})m d\ell, \quad (2)$$

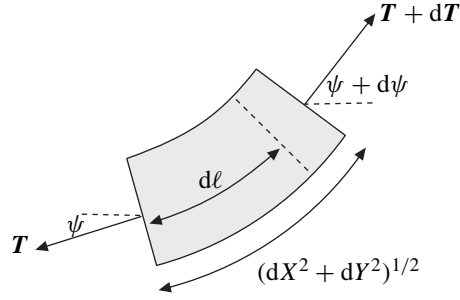


Figure 2: Sketch of cable element stretched under tension

where  $\{\ddot{\cdot}\}$  denotes a second derivative with respect to time. When we introduce

$$\frac{\partial X}{\partial \ell} = \left(1 + \frac{T}{EA}\right) \cos \psi, \quad \frac{\partial Y}{\partial \ell} = \left(1 + \frac{T}{EA}\right) \sin \psi, \quad (3)$$

we obtain [3] finally for  $d\ell \rightarrow 0$

$$\frac{\partial}{\partial \ell} \left( \frac{T}{1 + T/EA} \frac{\partial X}{\partial \ell} \right) = m \frac{\partial^2 X}{\partial t^2}, \quad (4a)$$

$$\frac{\partial}{\partial \ell} \left( \frac{T}{1 + T/EA} \frac{\partial Y}{\partial \ell} \right) = m \frac{\partial^2 Y}{\partial t^2} + mg, \quad (4b)$$

$$\left( \frac{\partial X}{\partial \ell} \right)^2 + \left( \frac{\partial Y}{\partial \ell} \right)^2 = \left(1 + \frac{T}{EA}\right)^2. \quad (4c)$$

The boundary conditions at the fixed supports at  $\ell = 0$  and  $\ell = L$  are (Figure 1)

$$X = 0, Y = 0 \quad (\ell = 0), \quad (5a)$$

$$X = S, Y = 0 \quad (\ell = L), \quad (5b)$$

## 2.2 Scaling, small parameters, and the mode of vibration

The type of motion of interest allows further reduction of the model, and is specified by

- The ratio of sag  $D$  and cable length  $L$  is small (typically  $\sim \frac{1}{30}$ ), so

$$\varepsilon = D/L \ll 1. \quad (6)$$

- The vertical displacement is of the order of the sag, so

$$Y/L = \mathcal{O}(\varepsilon). \quad (7)$$

- The transversal wave length  $\lambda_T$  is of the order of  $L$ , so

$$\lambda_T/L = \mathcal{O}(1). \quad (8)$$

- The relative stiffness  $EA/mgL$  is high, such that the longitudinal wave length  $\lambda_L$  is large compared to  $L$ . More specifically, we assume

$$L/\lambda_L = \mathcal{O}(\varepsilon). \quad (9)$$

- The relative amplitude  $\delta$  of the time dependent perturbation will be taken small but bigger than the orders of  $\varepsilon$  neglected. We will assume a nearly harmonic vibration, with a single dominating frequency  $\omega$ , which is to be found. Since we are interested in the intrinsic non-linear interaction between the harmonics, we will analyse the generated higher harmonics by a Lindstedt-Poincaré series in  $\delta$  [18–20]. To keep the results clear, higher harmonics not generated by the first one will be excluded.

### 2.3 Reduced problem

The small parameter  $\varepsilon$  will be used to reduce the above general problem to an asymptotic model. This model will then be analysed asymptotically for small relative amplitude  $\delta$ .

Since the longitudinal wave speed is  $c_L = (EA/m)^{1/2}$ , while  $L/\lambda_L = \omega L/c_L = \mathcal{O}(\varepsilon)$ , we introduce the reference frequency  $\omega_{\text{ref}} = \varepsilon(EA/m)^{1/2}/L$ . The dimensionless frequency and time variable are then given by

$$\omega = \omega_{\text{ref}} \omega^*, \quad t = t^*/\omega_{\text{ref}}. \quad (10)$$

Arc length  $\ell$  scales on  $L$ . Since  $Y/L = \mathcal{O}(\varepsilon)$ ,  $Y$  scales on  $\varepsilon L$ . The transversal wave velocity is  $c_T = (T/m)^{1/2}$ , so  $\lambda_T/L = c_T/\omega L = \mathcal{O}(1)$  yields that the tension scales on  $T_{\text{ref}} = \varepsilon^2 EA$ . Together we have for  $s \in [0, 1]$

$$\ell = sL, \quad Y(\ell, t) = \varepsilon LY^*(s, t^*), \quad T(\ell, t) = T_{\text{ref}} T^*(s, t^*). \quad (11)$$

By substituting above estimates in equation (4c), it transpires that  $\frac{\partial}{\partial \ell} X = 1 + \mathcal{O}(\varepsilon^2)$ , and so we write for  $X$

$$X(\ell, t) = Ls + \varepsilon^2 LX^*(s, t^*). \quad (12)$$

If we substitute the present estimates into equation (4b), we find the term  $mgL/EA\varepsilon^3$  next to terms of  $\mathcal{O}(1)$ . So we introduce

$$\mu = \frac{mgL}{8EA\varepsilon^3} = \mathcal{O}(1) \quad (13)$$

(where the factor 8 is included for notational convenience; *cf.* [3]). While we omit the superscript asterisks from here on, we obtain, under the approximation for small  $\varepsilon$ , the reduced version of problem (4) as follows

$$\frac{\partial T}{\partial s} = 0, \quad (14a)$$

$$\frac{\partial}{\partial s} \left( T \frac{\partial Y}{\partial s} \right) = 8\mu + \frac{\partial^2 Y}{\partial t^2}, \quad (14b)$$

$$\frac{\partial X}{\partial s} + \frac{1}{2} \left( \frac{\partial Y}{\partial s} \right)^2 = T. \quad (14c)$$

If sag  $D$  and span size  $S$  are known, cable length  $L$  – and hence  $\varepsilon$  and  $\mu$  – are to be determined from the stationary solution. We split up the solution in a stationary and non-stationary part

$$\begin{aligned} X(s, t) &= X_0(s) + x(s, t), \\ Y(s, t) &= Y_0(s) + y(s, t), \\ T(s, t) &= T_0(s) + \tau(s, t), \end{aligned} \quad (15)$$

with boundary conditions for the stationary part:  $Y_0(0) = Y_0(1) = X_0(0) = 0$ . From symmetry, the location of maximum deflection is halfway, so  $Y_0(\frac{1}{2}) = -1$ . The value  $X_0(1) = S_0$  determines the unknown length  $L$  by the relation  $S = L(1 + \varepsilon^2 S_0)$ . The stationary solution is easily found and given by

$$T_0(s) = \mu, \quad (16a)$$

$$Y_0(s) = -4(s - \frac{1}{2})^2, \quad (16b)$$

$$X_0(s) = \mu s - \frac{4}{3} - \frac{32}{3} (s - \frac{1}{2})^3, \quad (16c)$$

$$S_0 = \mu - \frac{8}{3}. \quad (16d)$$

If we substitute (15) and (16) into (14) we obtain our fundamental non-stationary problem

$$\frac{\partial \tau}{\partial s} = 0, \quad (17a)$$

$$(\mu + \tau) \frac{\partial^2 y}{\partial s^2} + 8\tau = \frac{\partial^2 y}{\partial t^2}, \quad (17b)$$

$$\frac{\partial x}{\partial s} + 8(s - \frac{1}{2}) \frac{\partial y}{\partial s} + \frac{1}{2} \left( \frac{\partial y}{\partial s} \right)^2 = \tau. \quad (17c)$$

The boundary conditions at the rigid supports follow readily:

$$x(0, t) = x(1, t) = y(0, t) = y(1, t) = 0. \quad (18)$$

### 3. LINDSTEDT-POINCARÉ ANALYSIS

We start with summarising the non-resonant results of [12].

#### 3.1 Away from resonance

By noting that

$$x(s, t) = \tau s - \frac{1}{2} \int_0^s (y_s)^2 ds - 8(s - \frac{1}{2})y(s, t) + 8 \int_0^s y ds, \quad (19a)$$

$$\tau(t) = \frac{1}{2} \int_0^1 (y_s)^2 ds - 8 \int_0^1 y ds, \quad (19b)$$

we can eliminate  $x$  and  $\tau$  to get

$$\mu \frac{\partial^2 y}{\partial s^2} - \frac{\partial^2 y}{\partial t^2} = \left( 8 + \frac{\partial^2 y}{\partial s^2} \right) \left( 8 \int_0^1 y ds - \frac{1}{2} \int_0^1 (y_s)^2 ds \right). \quad (20)$$

For the Lindstedt-Poincaré method we scale time on the (unknown) period  $2\pi/\omega$ . So we assume

$$t' = \omega t, \quad (21)$$

such that

$$\begin{aligned} \mu \frac{\partial^2 y}{\partial s^2} - \omega^2 \frac{\partial^2 y}{\partial t'^2} &= \left( 8 + \frac{\partial^2 y}{\partial s^2} \right) \left( 8 \int_0^1 y ds - \frac{1}{2} \int_0^1 (y_s)^2 ds \right) \\ &= 64 \int_0^1 y ds - 4 \int_0^1 (y_s)^2 ds + 8 \frac{\partial^2 y}{\partial s^2} \int_0^1 y ds - \frac{1}{2} \frac{\partial^2 y}{\partial s^2} \int_0^1 (y_s)^2 ds. \end{aligned}$$

We are interested, for small  $\delta$ , in the weakly nonlinear modes of “gravity” type, *i.e.* where  $\tau$  vanishes to leading order (Figure 3).

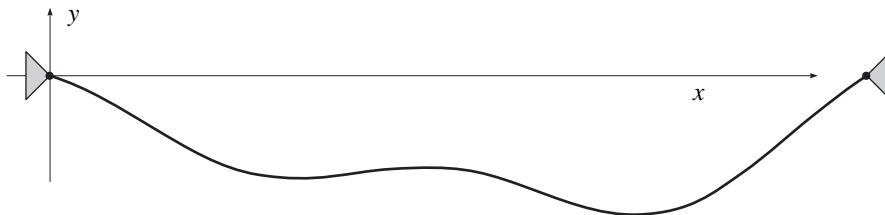


Figure 3: Sketch of cable gravity mode

This is found for

$$y = \delta y_1(s, t') + \delta^2 y_2(s, t') + \dots, \quad \omega = \omega_0 + \delta \omega_1 + \dots \quad (22)$$

with

$$y_1 = y_{11} \sin t' = \sin ks \sin t', \quad k = 2\pi n \quad (n \in \mathbb{N}), \quad \omega_0 = k\sqrt{\mu}, \quad (23)$$

such that  $\int_0^1 y_1 ds = 0$ , and therefore  $\tau = \mathcal{O}(\delta^2)$  because

$$\int_0^1 y ds = \mathcal{O}(\delta^2), \quad \int_0^1 (y_s)^2 ds = \mathcal{O}(\delta^2).$$

We obtain for  $y_2$

$$\mu \frac{\partial^2 y_2}{\partial s^2} - \omega_0^2 \frac{\partial^2 y_2}{\partial t'^2} = -2\omega_0 \omega_1 \sin ks \sin t' + 64 \int_0^1 y_2 ds - 2k^2 \left( \frac{1}{2} - \frac{1}{2} \cos 2t' \right).$$

There are no secular terms generated by  $\sin t'$  other than by the term  $\sim \omega_1$ , so we must choose  $\omega_1 = 0$ . We have only a constant and a second harmonic driving  $y_2$ , so we have to assume

$$y_2(s, t') = y_{20}(s) + y_{22}(s) \cos 2t', \quad (24a)$$

with

$$\begin{aligned} \mu \frac{d^2 y_{20}}{ds^2} &= 64 \int_0^1 y_{20} ds - k^2, \\ \mu \frac{d^2 y_{22}}{ds^2} + 4\omega_0^2 y_{22} &= 64 \int_0^1 y_{22} ds + k^2, \end{aligned}$$

with solutions (note that there is no need for a solution  $\sim \sin 2ks$  in  $y_{22}$ , so this is assumed to be not excited)

$$y_{20} = \frac{\frac{3}{2}k^2}{3\mu + 16} (s - s^2), \quad (24b)$$

$$y_{22} = -\frac{\frac{1}{4}k^2}{16 - \omega_0^2} (1 - \cos 2ks). \quad (24c)$$

For completeness we summarise the rest of the solution. For tension

$$\tau(t) = \delta^2 (\tau_{20} + \tau_{22} \cos 2t') + \dots \quad (25a)$$

we have

$$\tau_{20} = \frac{\frac{3}{8}\omega_0^2}{3\mu + 16}, \quad (25b)$$

$$\tau_{22} = \frac{\frac{1}{8}k^2\omega_0^2}{16 - \omega_0^2}. \quad (25c)$$

For axial displacement

$$x(s, t) = \delta x_{11} \sin t' + \delta^2 (x_{20} + x_{22} \cos 2t') + \dots \quad (26a)$$

we have

$$x_{11} = -8(s - \frac{1}{2}) \sin ks + 8 \frac{1 - \cos ks}{k}, \quad (26b)$$

$$x_{20} = -\frac{8k^2}{3\mu + 16}(s - s^2)(s - \frac{1}{2}) - \frac{1}{16}k \sin 2ks, \quad (26c)$$

$$x_{22} = \frac{1}{16}k \frac{\omega_0^2 - 32}{\omega_0^2 - 16} \sin 2ks - \frac{2k^2}{\omega_0^2 - 16}(s - \frac{1}{2})(1 - \cos 2ks). \quad (26d)$$

We observe that solution  $y_{22}$  breaks down if  $\omega_0 = 4$  or  $\mu = 16/k^2 = 4/(n\pi)^2 \simeq 0.405/n^2$ . In other words, there is a sequence of parameter values where the first and second harmonic are in resonance. For such a value the function  $\eta(s) = 1 - \cos 2ks$  becomes an eigensolution of the system for  $y_{22}$  and therefore satisfies the *unforced* equation

$$\mu \frac{d^2 \eta}{ds^2} + 4\omega_0^2 \eta = 64 \int_0^1 \eta ds.$$

Of course, the solution  $y$  will in reality not blow up, because for  $\mu k^2 = 16 + \mathcal{O}(\delta)$  the asymptotic hierarchy of the assumed  $\delta$ -expansion is destroyed and will have to be replaced by another expansion. In the following section we will analyse this resonance in more detail.

### 3.2 Near resonance

We assume from now on that

$$\mu = \frac{16}{k^2} + \alpha \delta \quad \text{and} \quad \omega = \omega_0 + \delta \omega_1 + \dots, \quad \text{where} \quad k = 2\pi n \quad \text{and} \quad \omega_0 = 4, \quad (27)$$

and study the same equation for similar solutions.

$$\left(\frac{16}{k^2} + \alpha \delta\right) \frac{\partial^2 y}{\partial s^2} - \omega^2 \frac{\partial^2 y}{\partial t'^2} = 64 \int_0^1 y ds - 4 \int_0^1 (y_s)^2 ds + 8 \frac{\partial^2 y}{\partial s^2} \int_0^1 y ds - \frac{1}{2} \frac{\partial^2 y}{\partial s^2} \int_0^1 (y_s)^2 ds.$$

We are interested in the modes

$$y = \delta y_1(s, t') + \delta^2 y_2(s, t') + \dots, \quad (28)$$

of the foregoing type, but now we have to add to  $y_1$  a multiple of  $(1 - \cos 2ks) \cos 2t'$ , because this term is excited at resonance, via the 2nd harmonic terms of 2nd order, and is therefore present at 1st order, albeit at as yet unknown amplitude. In contrast to the non-resonant case, it is now possible to add this term because it has become an eigensolution of the linear, *i.e.*  $\mathcal{O}(\delta)$ , equation. We therefore assume

$$y_1 = y_{11} \sin t' + y_{12} \cos 2t' = \sin ks \sin t' + A(1 - \cos 2ks) \cos 2t', \quad (29)$$

with  $k = 2\pi n$  and  $\omega_0 = 4$  and  $A$  to be determined. Since

$$\int_0^1 y_1 ds = \int_0^1 \sin ks ds \sin t' + A \int_0^1 (1 - \cos 2ks) ds \cos 2t' = A \cos 2t'$$

we obtain for  $y_2$

$$\frac{16}{k^2} \frac{\partial^2 y_2}{\partial s^2} - \omega_0^2 \frac{\partial^2 y_2}{\partial t'^2} = -\alpha \frac{\partial^2 y_1}{\partial s^2} + 2\omega_0 \omega_1 \ddot{y}_1 + 64 \int_0^1 y_2 ds - 4 \int_0^1 (y_1')^2 ds + 8y_1'' \int_0^1 y_1 ds$$

$$\begin{aligned}
&= 64 \int_0^1 y_2 \, ds - (\alpha y''_{11} + 2\omega_0\omega_1 y_{11}) \sin t' - (\alpha y''_{12} + 8\omega_0\omega_1 y_{12}) \cos 2t' \\
&\quad - 4 \int_0^1 \left[ (y'_{11})^2 (\sin t')^2 + 2y'_{11}y'_{12} \sin t' \cos 2t' + (y'_{12})^2 (\cos 2t')^2 \right] ds \\
&\quad\quad\quad + 8Ay''_{11} \sin t' \cos 2t' + 8Ay''_{12} (\cos 2t')^2 \\
&= 64 \int_0^1 y_2 \, ds - k^2 - 4A^2k^2(1 - 4 \cos 2ks) \\
&\quad + (\alpha k^2 + 4Ak^2 - 2\omega_0\omega_1) \sin ks \sin t' \\
&\quad + (k^2 - 4\alpha Ak^2 \cos 2ks - 8A\omega_0\omega_1(1 - \cos 2ks)) \cos 2t' \\
&\quad - 4Ak^2 \sin ks \sin 3t' - 4A^2k^2(1 - 4 \cos 2ks) \cos 4t'.
\end{aligned}$$

From the forcing term it follows that a steady solution has to be of the following form

$$y_2(s, t') = y_{20}(s) + y_{21}(s) \sin t' + y_{22}(s) \cos 2t' + y_{23}(s) \sin 3t' + y_{24}(s) \cos 4t', \quad (30a)$$

so that we have per harmonic the equations

$$\begin{aligned}
\frac{16}{k^2} y''_{20} &= 64 \int_0^1 y_{20} \, ds - k^2 - 4A^2k^2(1 - 4 \cos 2ks), \\
\frac{16}{k^2} y''_{21} + \omega_0^2 y_{21} &= 64 \int_0^1 y_{21} \, ds + (\alpha k^2 + 4Ak^2 - 2\omega_0\omega_1) \sin ks, \\
\frac{16}{k^2} y''_{22} + 4\omega_0^2 y_{22} &= 64 \int_0^1 y_{22} \, ds + k^2 - 4\alpha Ak^2 \cos 2ks - 8A\omega_0\omega_1(1 - \cos 2ks), \\
\frac{16}{k^2} y''_{23} + 9\omega_0^2 y_{23} &= 64 \int_0^1 y_{23} \, ds - 4Ak^2 \sin ks, \\
\frac{16}{k^2} y''_{24} + 16\omega_0^2 y_{24} &= 64 \int_0^1 y_{24} \, ds - 4A^2k^2(1 - 4 \cos 2ks),
\end{aligned}$$

and boundary conditions  $y_2(0) = y_2(1) = 0$ . The solutions are found to be

$$y_{20} = -\frac{9}{8}k^4 \frac{A^2 - \frac{1}{12}}{k^2 + 3} (s - s^2) + \frac{1}{4}A^2k^2(1 - \cos 2ks), \quad (30b)$$

$$y_{21} = C_{21} \sin ks, \quad (30c)$$

$$y_{22} = \frac{1}{4}A^2k^3(s - \frac{1}{2}) \sin 2ks + C_{22}^{(1)} \sin 2ks + C_{22}^{(2)}(1 - \cos 2ks), \quad (30d)$$

$$y_{23} = -\frac{1}{32}Ak^2 \sin ks + C_{23} \sin 3ks, \quad (30e)$$

$$y_{24} = -\frac{1}{12}A^2k^2(1 - \cos 2ks) + \frac{1}{16}A^2k^2(1 - \cos 4ks) + C_{24} \sin 4ks, \quad (30f)$$

under the solvability conditions that

$$\omega_1 = \left(\frac{1}{2}A + \frac{1}{8}\alpha\right) k^2, \quad (31)$$

while  $A$  is the solution of

$$24A^2 + 4\alpha A = 1. \quad (32)$$

We consider only the causal solution  $y_2$  and suppress the undriven and therefore unnecessary multiples of  $\sin ks$ ,  $\sin 2ks$ ,  $(1 - \cos 2ks)$ ,  $\sin 3ks$ , and  $\sin 4ks$ . This means that we assume  $C_{21} = C_{22}^{(1)} = C_{22}^{(2)} = C_{23} = C_{24} = 0$ . Only for esthetic reasons of symmetry, we retained the factor  $(s - \frac{1}{2})$  in  $y_{22}$ , rather than simply  $s$ .



### 3.3 Matching with the non-resonant solution

There are two branches for  $A$ , namely

$$A_{\pm} = -\frac{1}{12}\alpha \pm \frac{1}{12}\sqrt{6 + \alpha^2}. \quad (33)$$

When  $\alpha$  is taken large positive or negative, each branch matches to the non-resonant solution in one direction, but not in the other. This is seen as follows.

$A_+$  is everywhere positive and tends to zero for  $\alpha \rightarrow \infty$  and diverges linearly to infinity for  $\alpha \rightarrow -\infty$ .  $A_-$  is just the opposite: everywhere negative and tends to zero for  $\alpha \rightarrow -\infty$  and diverges for  $\alpha \rightarrow \infty$ . In order to match with the non-resonant solution (where  $A = 0$ ) we have to choose for large  $\alpha$  the decaying solutions, *viz.*  $A = A_+$  if  $\alpha \rightarrow \infty$  and  $A = A_-$  if  $\alpha \rightarrow -\infty$ . Indeed, this gives both the amplitude of  $y_{22}$  in [24c]

$$A \simeq \frac{1}{4\alpha} = \frac{\frac{1}{4}k^2\delta}{\alpha\delta k^2} = -\frac{\frac{1}{4}k^2\delta}{16 - \mu k^2}$$

and the matching value of  $\omega$

$$\omega_0 + \delta\omega_1 = 4 + \frac{1}{8}\alpha k^2\delta \simeq \sqrt{16 + \alpha\delta k^2} = \sqrt{k^2\mu} = \omega.$$

We have not investigated here where and how  $A$  changes from one branch to the other when  $\alpha$  is varied from large negative to large positive.

### 3.4 The corresponding tension and $x$ -deflection

The corresponding tension  $\tau$  is found to consist only of multiples of the 2nd harmonic, in contrast to  $y$  and  $x$ . We have

$$\begin{aligned} \tau &= -8\delta \int_0^1 y_1(s, t') ds - 8\delta^2 \int_0^1 y_2(s, t') ds + \frac{1}{2}\delta^2 \int_0^1 (y_{1s}(s, t'))^2 ds + \dots \\ &= -8\delta A \cos 2t' - \frac{9}{2}\delta^2 k^2 \frac{A^2 - \frac{1}{12}}{k^2 + 3} + \delta^2 k^2 (A^2 - \frac{1}{8}) \cos 2t' + \frac{2}{3}\delta^2 k^2 A^2 \cos 4t' + \dots \end{aligned} \quad (34)$$

which amounts to

$$\tau = \delta\tau_1 + \delta^2\tau_2 + \dots \quad (35)$$

with

$$\tau_1 = \tau_{12} \cos 2t', \quad (36a)$$

$$\tau_{12} = -8A, \quad (36b)$$

and

$$\tau_2 = \tau_{20} + \tau_{22} \cos 2t' + \tau_{24} \cos 4t', \quad (37a)$$

$$\tau_{20} = -\frac{9}{2}k^2 \frac{A^2 - \frac{1}{12}}{k^2 + 3}, \quad (37b)$$

$$\tau_{22} = k^2 (A^2 - \frac{1}{8}), \quad (37c)$$

$$\tau_{24} = \frac{2}{3}k^2 A^2. \quad (37d)$$

### 3.5 The corresponding $x$ -deflection

The corresponding  $x$ -deflection is given by

$$\begin{aligned}
 x &= \tau s - \frac{1}{2} \int_0^s (y_s)^2 ds - 8(s - \frac{1}{2})y + 8 \int_0^s y ds, \\
 &= \delta \tau_1 s + \delta^2 \tau_2 s - \frac{1}{2} \delta^2 \int_0^s (y_{1,s})^2 ds - 8(s - \frac{1}{2})(\delta y_1 + \delta^2 y_2) + 8\delta \int_0^s y_1 ds + 8\delta^2 \int_0^s y_2 ds, \\
 &= \delta x_1 + \delta^2 x_2
 \end{aligned} \tag{38}$$

where

$$\begin{aligned}
 x_1 &= \tau_1 s - 8(s - \frac{1}{2})y_1 + 8 \int_0^s y_1 ds, \\
 x_2 &= \tau_2 s - \frac{1}{2} \int_0^s (y_{1,s})^2 ds - 8(s - \frac{1}{2})y_2 + 8 \int_0^s y_2 ds.
 \end{aligned}$$

We obtain for  $x_1$

$$x_1 = x_{11} \sin t' + x_{12} \cos 2t' \tag{39a}$$

the equations

$$\begin{aligned}
 x_{11} &= -8(s - \frac{1}{2})y_{11} + 8 \int_0^s y_{11} ds, \\
 x_{12} &= \tau_{12} s - 8(s - \frac{1}{2})y_{12} + 8 \int_0^s y_{12} ds,
 \end{aligned}$$

with solutions

$$x_{11} = -8(s - \frac{1}{2}) \sin ks + 8 \frac{1 - \cos ks}{k}, \tag{39b}$$

$$x_{12} = -8As - 8A(s - \frac{1}{2})(1 - \cos 2ks) + 8A \left( s - \frac{\sin 2ks}{2k} \right). \tag{39c}$$

Similarly for  $x_2$

$$x_2 = x_{20} + x_{21} \sin t' + x_{22} \cos 2t' + x_{23} \sin 3t' + x_{24} \cos 4t' \tag{40a}$$

the equations

$$\begin{aligned}
 x_{20} &= \tau_{20} s - \frac{1}{4} \int_0^s (y'_{11})^2 ds - \frac{1}{4} \int_0^s (y'_{12})^2 ds - 8(s - \frac{1}{2})y_{20} + 8 \int_0^s y_{20} ds, \\
 x_{21} &= \frac{1}{2} \int_0^s y'_{11} y'_{12} ds, \\
 x_{22} &= \tau_{22} s + \frac{1}{4} \int_0^s (y'_{11})^2 ds - 8(s - \frac{1}{2})y_{22} + 8 \int_0^s y_{22} ds, \\
 x_{23} &= -\frac{1}{2} \int_0^s y'_{11} y'_{12} ds - 8(s - \frac{1}{2})y_{23} + 8 \int_0^s y_{23} ds, \\
 x_{24} &= \tau_{24} s - \frac{1}{4} \int_0^s (y'_{12})^2 ds - 8(s - \frac{1}{2})y_{24} + 8 \int_0^s y_{24} ds,
 \end{aligned}$$

with the results

$$x_{20} = 6k^4 \frac{A^2 - \frac{1}{12}}{k^2 + 3} (s - \frac{1}{2})(s - s^2) - 2A^2 k^2 (s - \frac{1}{2})(1 - \cos 2ks) - (A^2 + \frac{1}{16})k \sin 2ks + \frac{1}{8}A^2 k \sin 4ks, \quad (40b)$$

$$x_{21} = \frac{1}{2}kA(1 - \cos ks) + \frac{1}{6}kA(1 - \cos 3ks), \quad (40c)$$

$$x_{22} = k^2 A^2 (s - \frac{1}{2})(1 - \cos 2ks) + (\frac{1}{2}k(A^2 + \frac{1}{8}) - 2A^2 k^3 (s - \frac{1}{2})^2) \sin 2ks, \quad (40d)$$

$$x_{23} = \frac{1}{4}k^2 A (s - \frac{1}{2}) \sin ks - \frac{3}{4}kA(1 - \cos ks) - \frac{1}{6}kA(1 - \cos 3ks), \quad (40e)$$

$$x_{24} = \frac{1}{3}kA^2 \sin 2ks + \frac{2}{3}k^2 A^2 (s - \frac{1}{2})(1 - \cos 2ks) - \frac{1}{2}k^2 A^2 (s - \frac{1}{2})(1 - \cos 4ks). \quad (40f)$$

#### 4. EXAMPLES

To illustrate the present theory we have calculated the following example. The relevant dimensionless problem parameters are

$$\varepsilon = \frac{1}{30}, \quad \delta = 0.4, \quad k = 2\pi, \quad \alpha = 0, \quad A = \frac{1}{12}\sqrt{6}, \quad \mu = \frac{16}{k^2} = 0.4053, \quad \omega = 5.6117.$$

A possible dimensional realisation is

$$EA = 77000 \text{ N/mm}^2 \times 280 \text{ mm}^2 = 21.56 \cdot 10^6 \text{ N}, \quad S = 300 \text{ m}, \quad D = 10.025 \text{ m}, \\ mg = 0.878 \text{ kg/m} \times 9.8 \text{ m/s}^2 = 8.908 \text{ kg/s}^2, \quad L = 300.757 \text{ m},$$

and a frequency of 0.49 seconds. The dimensionless amplitude  $\delta$  is taken high enough to make the higher order effects visible. Otherwise it is the result of the excitation process not included in our model.

Shown are a full period of the varying dimensionless cable position, given by  $(s + \varepsilon^2 X_0 + \varepsilon^2 x, Y_0 + y)$ , in Figure (4) and the dimensionless total tension  $T_0 + \tau$  varying in time in Figure (5). The dotted line corresponds to the respective stationary part. It is clearly seen that the cable vibrates at two dominant frequencies at the same time.

Strictly speaking, amplitude  $\delta$  is too large to be realistic, as the total tension becomes negative for certain intervals (in reality this would make the cable slack, the tension  $\equiv 0$ , and the equations very different). We have retained the example, however, because the behaviour of the higher harmonics is better visible.

The example confirms that the intermodal resonance can occur at realistic parameter values.

#### 5. CONCLUSIONS

The problem of a single-span freely in-plane oscillating suspended cable is systematically analysed by means of a Lindstedt-Poincaré solution near a resonance between the first and second harmonic. This resonance occurs for a cable motion, where the tension (away from resonance) vanishes to leading order. This is called a gravity mode since elasticity is of minor importance (*cf.* an oscillating catenary). Because of this resonance, a strictly gravity type of motion is now impossible, and the tension-free first harmonic is always accompanied by a second harmonic with non-zero tension.

Since a galloping overhead transmission cable is excited by a combined effect of wind and a torsional motion of the cable in the same frequency as the vertical motion, the presence of the second harmonic due to resonance may have a stabilising effect. This second harmonic is not in phase with the torsional motion, and therefore not maintained by the wind. In fact, for a part of the cycle the effect of the wind may be counteractive and thus hindering the galloping motion. Therefore, designing overhead transmission cables at parameter values of the resonance may be favourable for suppressing galloping.

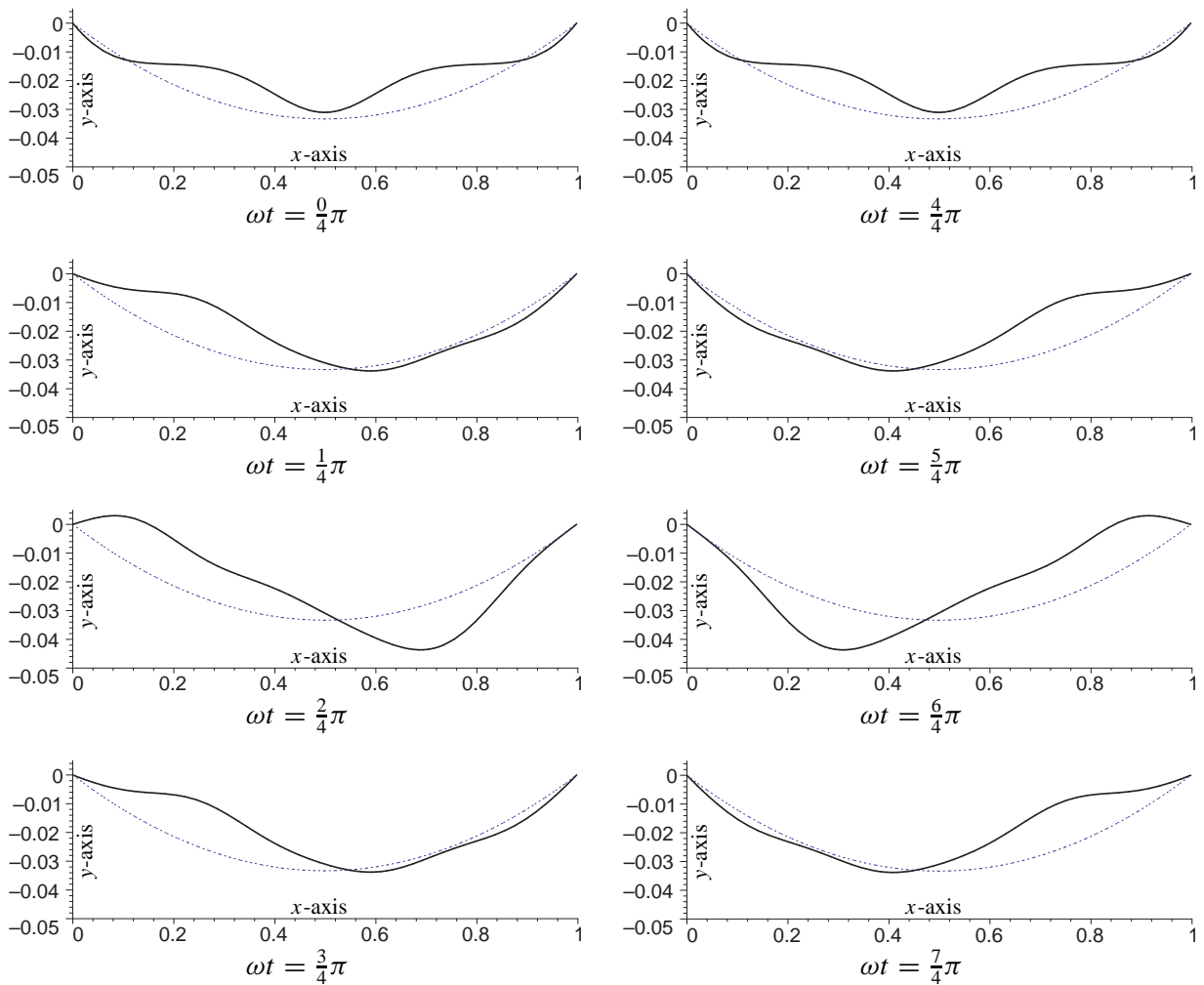


Figure 4: A full period of motion of  $(s + \varepsilon^2 X_0(s) + \varepsilon^2 x(s, t), Y_0(s) + y(s, t))$  for  $\varepsilon = \frac{1}{30}$ ,  $\delta = 0.4$ ,  $\alpha = 0$ ,  $k = 2\pi$ . The dotted line corresponds to the stationary part  $(X_0, Y_0)$ .

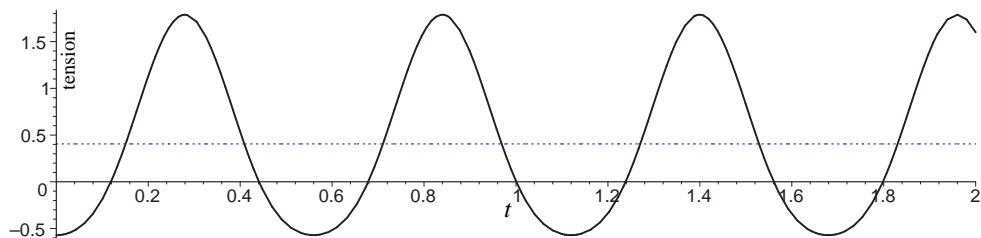


Figure 5: The corresponding tension  $T_0 + \tau(t)$ , where  $\omega = 5.61$ . The dotted line corresponds to the stationary part  $T_0$ . Strictly speaking, amplitude  $\delta$  is in the example too large to be realistic, as the tension becomes negative for certain intervals (in reality this would make the cable slack and the equations very different). We have retained the example, however, because the behaviour of the higher harmonics is better visible.

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