

# Impedance models in time domain

TU/e contribution to Deliverable 3.5.1 of Task 3.5

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## 1 Summary

Necessary conditions for an impedance function are derived. Methods available in the literature are discussed. A format with recipe is proposed for an exact impedance condition in time domain on a time grid, based on the Helmholtz resonator model. An explicit solution is given of a pulse reflecting in time domain at a Helmholtz resonator impedance wall.

## 2 Introduction

We use throughout the  $e^{+i\omega t}$  convention.

Acoustic liners are usually described in frequency domain by their impedance. If we use a numerical method in time domain, this is to be translated into an equivalent relation. This is not straightforward. For example, it is principally not possible for a non-real impedance to retain this value for all frequencies. We have to extend the impedance as a function in the complex frequency. This requires certain specific conditions. If these are not obeyed, the impedance would be physically impossible. This would manifest itself by, for example, non-physical instabilities.

It is remarkable that only relatively recently the frequency dependence of the liner was left unnoticed in a stability analysis in frequency domain [1].

## 3 The problem

A wall, applied with locally reacting lining of impedance type, is without mean flow described by

$$\hat{p}(0, \omega) = Z(\omega)\hat{v}(0, \omega), \quad (1)$$

where  $\hat{v} = (\hat{v} \cdot \mathbf{n})$ ,  $\hat{v}$  is the acoustic velocity vector and  $\mathbf{n}$  denotes the normal vector of the wall that points *into* the wall. Pressure is scaled on  $\rho_0 c_0^2$ , velocity on  $c_0$  and impedance on  $\rho_0 c_0$ . In time domain relation (1) becomes

$$p(0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(t - \tau)v(0, \tau) d\tau \quad (2)$$

where  $p(x, t)$  and  $v(x, t)$  denote the inverse Fourier transforms of  $\hat{p}(x, \omega)$  and  $\hat{v}(x, \omega)$ , respectively,

$$p(x, t) = \int_{-\infty}^{\infty} \hat{p}(0, \omega) e^{i\omega t} d\omega, \quad v(x, t) = \int_{-\infty}^{\infty} \hat{v}(0, \omega) e^{i\omega t} d\omega, \quad (3)$$

$z(t)$  denotes the inverse Fourier transform of  $Z(\omega)$

$$z(t) = \int_{-\infty}^{\infty} Z(\omega) e^{i\omega t} d\omega \quad (4)$$

and the integral over  $z(t - \tau)v(0, \tau)$  is called a convolution product.

Equivalent other forms are also possible. If we express  $v$  in  $p$  by using the admittance  $Y(\omega) = Z(\omega)^{-1}$ , we have

$$v(0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(t - \tau)p(0, \tau) d\tau \quad (5)$$

where  $y(t)$  denotes the inverse Fourier transform of  $Y(\omega)$ . If we write  $Z(\omega) = A(\omega)/B(\omega)$  and bring  $B$  to the left-hand side of (1), we obtain a convolution on both sides

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} b(t - \tau)p(0, \tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(t - \tau)v(0, \tau) d\tau \quad (6)$$

where  $a(t)$  and  $b(t)$  denote the inverse Fourier transforms of  $A(\omega)$  and  $B(\omega)$ .

With mean flow, we have (according to Myers [2] for a certain limit of mean flow boundary layer and acoustic boundary layer thickness being small compared to the acoustic wavelength)

$$(\hat{\mathbf{v}} \cdot \mathbf{n}) = (i\omega + \mathbf{v}_0 \cdot \nabla - \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \mathbf{v}_0)) \frac{\hat{p}}{i\omega Z}. \quad (7a)$$

As the mean flow usually satisfies  $(\mathbf{v}_0 \cdot \mathbf{n}) = 0$ , this may be recast into

$$(\hat{\mathbf{v}} \cdot \mathbf{n}) = (i\omega + \mathbf{v}_0 \cdot \nabla + \mathbf{v}_0 \cdot (\mathbf{n} \cdot \nabla \mathbf{n})) \frac{\hat{p}}{i\omega Z}, \quad (7b)$$

where, incidentally,  $(\mathbf{n} \cdot \nabla \mathbf{n})$  is tangential to the wall surface and vanishes for a flat surface. Other forms, that utilize  $\nabla \cdot (\rho_0 \mathbf{v}_0) = 0$ , are also possible [3]. If  $Z$  is independent of position we get

$$i\omega Z \hat{v} = i\omega \hat{p} + \mathbf{v}_0 \cdot \nabla \hat{p} + \mathbf{v}_0 \cdot (\mathbf{n} \cdot \nabla \mathbf{n}) \hat{p}, \quad (7c)$$

In time domain this becomes

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} z(t - \tau) \frac{\partial}{\partial \tau} v(0, \tau) d\tau = \frac{\partial}{\partial t} p + \mathbf{v}_0 \cdot \nabla p + \mathbf{v}_0 \cdot (\mathbf{n} \cdot \nabla \mathbf{n}) p, \quad (8)$$

or any other equivalent form.

When we are primarily interested in the frequency domain solution for a particular, given, frequency, detailed knowledge of  $Z(\omega)$  in frequency domain is usually both irrelevant and not available. On the other hand, if we aim at achieving this solution by means of a time dependent model, we need to construct for the design impedance  $Z_0$  at  $\omega = \omega_0$  a physically possible model  $Z(\omega)$  with the property  $Z(\omega_0) = Z_0$  that has an explicitly known and convenient inverse Fourier transform that can be used in our solution method.

In particular, if we assume that we have  $p$  and  $v$  available along the time grid  $t = n\Delta$  for  $n = 0, 1, \dots$ , the boundary condition in time domain should be easily expressible in terms of this discrete representation.

The purpose of the present document is to derive a class of impedance realisations that reduce in time domain to a relation between a limited number of the most recent terms of such time series.

## 4 Fundamental conditions

Since  $p(0, t)$  cannot depend on  $v(0, t)$  of the future, we must have  $z(t - \tau) = 0$  for  $\tau > t$  and  $Z(\omega)$  has to satisfy the causality condition (see Appendix 9)

$$Z(\omega) \text{ is analytic in } \text{Im}(\omega) < 0. \quad (9)$$

Since  $v(0, t)$  cannot depend on  $p(0, t)$  of the future, the admittance  $Z(\omega)^{-1}$  must be causal and thus

$$Z(\omega) \text{ is non-zero in } \text{Im}(\omega) < 0. \quad (10)$$

In a similar way any representation of the form  $Z(\omega) = A(\omega)/B(\omega)$  must be such that

$$A(\omega) \text{ and } B(\omega) \text{ are analytic in } \text{Im}(\omega) < 0. \quad (11)$$

Since  $p(0, t)$  and  $v(0, t)$  are real,  $z(t)$  has to be real too, and so  $Z(\omega)$  has to satisfy the reality condition

$$Z^*(\omega) = Z(-\omega), \quad (12)$$

where  $*$  denotes the complex conjugate. In other words, the real part  $\text{Re}(Z)$  (resistance) is even and the imaginary part  $\text{Im}(Z)$  (reactance) is odd in  $\omega$ .

Usually, the impedance wall is a passive wall that absorbs energy at any frequency, *i.e.*, the acoustic intensity into the wall  $\mathcal{I} = \frac{1}{2} \text{Re}(\hat{p}\hat{v}^*) = \frac{1}{2} \text{Re}(Z)|\hat{v}|^2$  is positive. As a result, the resistance  $\text{Re}(Z)$  has to be positive. In other words, the impedance satisfies the condition of passivity

$$\text{Re}(Z(\omega)) \geq 0 \quad \text{for all } \omega \in \mathbb{R}. \quad (13)$$

It should be noted that violation of the reality condition produces physically impossible results; violation of causality produces instabilities due “decay” in reversed time; violating the passivity condition implies a wall that produces instead of absorbs energy which will also lead to an instability. Next to these unphysical instabilities which are artefacts due to bad modelling, there is apparently with mean flow the possibility of a physical instability, comparable with a Kelvin-Helmholtz instability [4]. This potential instability is physical and cannot be ignored, although its role may be overestimated in the present linearised flow models.

Altogether we have the following conditions on  $Z$ :

### Theorem 1

*Conditions for a physically representable impedance  $Z(\omega)$  are*

1.  $Z(\omega)$  is analytic in  $\text{Im}(\omega) < 0$ ,
2.  $Z(\omega)$  is non-zero in  $\text{Im}(\omega) < 0$ ,
3.  $Z(\omega) = Z^*(-\omega)$  for any  $\omega \in \mathbb{R}$ ,
4.  $\text{Re}(Z(\omega)) \geq 0$  for any  $\omega \in \mathbb{R}$ .

If we want to split  $Z$  into a denominator/nominator, we have the following condition

### Theorem 2

*We may Fourier transform the relation  $B(\omega)\hat{p}(\omega) = A(\omega)\hat{v}(\omega)$  back into time domain, with  $Z(\omega) = A(\omega)/B(\omega)$ , only if  $A(\omega)$  and  $B(\omega)$  are analytic in  $\text{Im}(\omega) < 0$ .*

## 4.1 Examples

### 4.1.1 Mass-spring-damper

The mass-spring-damper model, given by

$$Z(\omega) = R + i\omega m - \frac{iK}{\omega} \quad (14)$$

must have  $R \geq 0$ , and has a pole at  $\omega = 0$ , which should be accounted to the upper half plane, while the zeros are located in the upper half plane as follows:

- if  $R = 0$ , then  $\omega = \pm\sqrt{K/m}$ , and  $K/m > 0$  ;
- if  $m = 0$ , then  $\omega = iK/R$ , and  $K > 0$  ;
- if  $K = 0$ , then  $\omega = iR/m$ , and  $m > 0$  ;
- if else, then  $\omega = i(2m)^{-1}R(1 \pm \sqrt{1 - 4mK/R^2})$ , and  $m > 0$  .

In other words,  $Z$  satisfies all conditions of theorem 1 if all parameters are interpreted physically and considered only for non-negative values. The corresponding inverse Fourier transform is

$$\frac{z(t)}{2\pi} = R\delta(t) + mv'(t) + KH(t) \quad (15)$$

leading to

$$p(t) = Rv(t) + mv'(t) + K \int_{-\infty}^t v(\tau) d\tau. \quad (16)$$

If we write

$$Z(\omega) = \frac{i\omega R + (i\omega)^2 m + K}{i\omega} \quad (17)$$

we obtain

$$p'(t) = Rv'(t) + mv''(t) + Kv(t). \quad (18)$$

### 4.1.2 A rational function in $\omega$

A useful generalisation of the previous model is an impedance written as a rational function in  $\omega$ , such that any zero's and poles are located in the upper half plane<sup>1</sup>. To enforce this causality condition is not very difficult, but it is less easy to formulate a general statement how to make sure that the conditions of reality and passivity are satisfied.

For real and positive  $q, a, b, c$ , with  $2b > c$ , we have the following simple form

$$Z(\omega) = iq \frac{(\omega - a - ib)(\omega + a - ib)}{\omega - ic} \quad (19)$$

with

$$z(t) = 2\pi q(a^2 + (b - c)^2)H(t) e^{-ct}, \quad (20)$$

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<sup>1</sup>Note that this is often confused with a Padé approximation. For any given (smooth) function, the Padé approximation is a uniquely defined best approximation by a rational function, without any further control of the zero's and poles. So for clarity we will avoid the word "Padé approximation" here.

that satisfies all conditions of causality, reality, and passivity. In time domain it corresponds with the condition

$$p(t) = q(a^2 + (b - c)^2) \int_{-\infty}^t e^{-c\tau} v(\tau) d\tau \quad (21)$$

or

$$cp(t) + p'(t) = q(a^2 + b^2)v(t) + 2bqv'(t) + qv''(t). \quad (22)$$

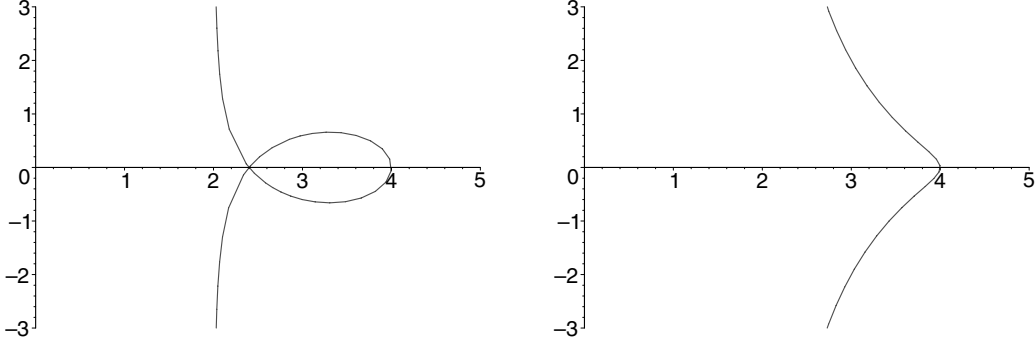


Figure 1: Left:  $a = 1, b = 3, c = 1, q = \frac{2}{5}$ , right:  $a = \frac{1}{5}\sqrt{23}, b = 1, c = \frac{6}{5}, q = \frac{5}{2}$ .

The impedance model has the following properties (see figure 1):

- $\text{Re}(Z(\omega)) \rightarrow q(2b - c)$  for  $\omega \rightarrow \pm \infty$ .
- $Z(0) = q(a^2 + b^2)/c$
- $\text{Im}(Z) = 0$  for  $\omega = 0$  and, if the square root exists, for  $\omega = \pm\sqrt{a^2 + b^2 - 2bc}$ .
- If  $a^2 + b^2 - 2bc < 0$ ,  $\text{Im}(Z) > 0$  for  $\omega > 0$  and  $\text{Im}(Z) < 0$  for  $\omega < 0$ .
- If  $a^2 + b^2 - 2bc > 0$ ,  $\text{Im}(Z)$  attains both positive and negative values for  $\omega > 0$ , by a loop where the curve crosses itself at  $Z = 2qb$
- If  $a^2 + b^2 - 2bc = 0$ , the curve passes a cusp singularity.

### 4.1.3 Basic Helmholtz Resonator

A very important example is the basic Helmholtz resonator model given by

$$Z(\omega) = R + i\omega m - i \cot(\omega L/c_0), \quad (23)$$

which is a good representation of a wall consisting of an array of Helmholtz resonators (see figure 2) with face-sheet resistance  $R$ , face-sheet mass reactance  $\omega m$ , cavity reactance  $-i \cot(\omega L/c_0)$ , cell depth  $L$  and sound speed  $c_0$ . It is straightforward that  $Z(\omega) = Z(-\omega)^*$ . With  $R, m, L, c_0 \geq 0$ ,  $\text{Re}(Z) = R \geq 0$  is trivially satisfied, while  $Z(\omega)$  is free of zeros in the lower half plane: write  $\omega = \xi - i\eta$ , then for  $\eta > 0$

$$\text{Re}(Z) = R + m\eta + \frac{\sinh(2\eta L/c_0)}{\cosh(2\eta L/c_0) - \cos(2\xi L/c_0)} > 0 \quad (24)$$

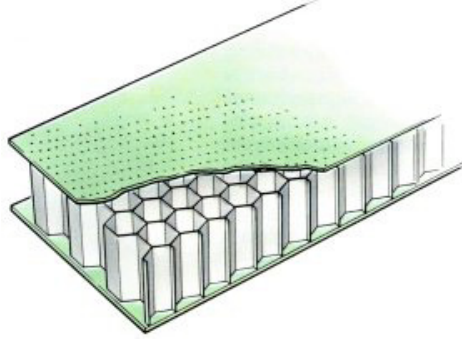


Figure 2: An acoustic liner of Helmholtz resonators.

so  $Z$  cannot be zero.  $Z$ 's real poles at  $\omega = n\pi c_0/L$  ( $n \in \mathbb{Z}$ ) are to be accounted to the upper half plane. So for  $\text{Im}(\omega) < 0$

$$\begin{aligned}
 Z(\omega) &= R + i\omega m - i \cot(\omega L/c_0) \\
 &= R + i\omega m + \frac{e^{i\omega L/c_0} + e^{-i\omega L/c_0}}{e^{i\omega L/c_0} - e^{-i\omega L/c_0}} \\
 &= R + i\omega m + 1 + 2 \frac{e^{-2i\omega L/c_0}}{1 - e^{-2i\omega L/c_0}} \\
 &= R + i\omega m + 1 + 2 \sum_{n=1}^{\infty} e^{-2in\omega L/c_0},
 \end{aligned} \tag{25}$$

and we have for the inverse Fourier transform the  $\delta$ -comb [5]

$$\frac{z(t)}{2\pi} = R\delta(t) + m\delta'(t) + \delta(t) + 2 \sum_{n=1}^{\infty} \delta(t - \lambda_n), \quad \lambda_n = \frac{2nL}{c_0}. \tag{26}$$

This yields in time domain the condition

$$p(t) = Rv(t) + mv'(t) + v(t) + 2 \sum_{n=1}^{\infty} v(t - \lambda_n). \tag{27}$$

Instead of expanding the geometrical series in  $e^{-2i\omega L/c_0}$  we may also write

$$Z(\omega) = \frac{(R + i\omega m)(1 - e^{-2i\omega L/c_0}) + 1 + e^{-2i\omega L/c_0}}{1 - e^{-2i\omega L/c_0}} \tag{28}$$

to arrive at the shorter alternative (without the infinite series)

$$p(t) - p(t - \lambda_1) = R(v(t) - v(t - \lambda_1)) + m(v'(t) - v'(t - \lambda_1)) + v(t) + v(t - \lambda_1). \tag{29}$$

#### 4.1.4 Double layer of Helmholtz resonators

An interesting generalisation that is worthy of further research is the configuration of two Helmholtz resonators on top of each other, separated by a septum [8]. If the top cavity has depth  $L_1$ , the bottom

cavity has depth  $L_2$ , while  $L = L_1 + L_2$ , and the face sheet impedance is given by  $Z_1 = R_1 + i\omega m_1$  and the septum impedance is  $Z_2 = R_2 + i\omega m_2$ , then we have for the total impedance

$$Z = Z_1 + \frac{Z_2 \frac{\cos(\omega L_1/c_0) \sin(\omega L_2/c_0)}{\sin(\omega L/c_0)} - i \cot(\omega L/c_0)}{1 + iZ_2 \frac{\sin(\omega L_1/c_0) \sin(\omega L_2/c_0)}{\sin(\omega L/c_0)}} \quad (30)$$

From its construction we may expect that this impedance function will satisfy all necessary conditions if the parameters are assumed positive. This is, however, difficult to prove. For example, an exhaustive study of all poles and zeros is very difficult for general cell depths  $L_1$  and  $L_2$ . It is, however, really feasible for simpler cases like when  $m_1 = m_2 = 0$  and  $L_1 = L_2$ .

## 5 Physically inspired forms

One might be tempted to believe that impedance models which are directly taken from representations of physical realisations would automatically satisfy all the necessary conditions. In some cases this may be true, but in general the expressions are either too complicated to be useful, or simplified or approximated to such an extent that the fundamental conditions are violated.

For example, we find in [6, 7, 8] impedance models of the form

$$Z = -\frac{i\omega\tau}{\sigma c_0} \left[ \frac{2J_1(K)}{K J_0(K)} - 1 \right] + \frac{1}{\sigma} \left\{ \left[ 1 - \frac{2}{\gamma} J_1(\gamma) \right] + iM_1(\gamma) \right\} - i \cot\left(\frac{\omega L}{c_0}\right) \quad (31)$$

where  $J_n$  denotes an  $n$ -th order Besselfunction [9],  $d$  the hole diameter,  $\tau$  the plate thickness,  $\nu$  the air kinematic viscosity and  $\sigma$  is the porosity (fraction of open area). The parameters  $K$ ,  $\gamma$  and  $M_1$  are defined by

$$K^2 = -\frac{i\omega d^2}{4\nu}, \quad \gamma = \frac{\omega d}{c_0}, \quad M_1(\gamma) = \frac{4}{\pi} \int_0^{\pi/2} \sin(\gamma \cos \alpha) \sin(\alpha)^2 d\alpha.$$

Although the expression appears to be analytic in a half plane, little can be said about the location of the zeros, while the construction of the inverse Fourier transform is not easy. Since in most aero-engine applications  $|K|$  is large and  $\gamma$  is small, the following approximation is used

$$Z = (1 + i) \frac{\sqrt{8\omega\nu}}{\sigma c_0} \left( 1 + \frac{\tau}{d} \right) + \frac{\omega^2 d^2}{8\sigma c_0^2} + \frac{i\omega\tau}{\sigma c_0} - i \cot\left(\frac{\omega L}{c_0}\right), \quad (32)$$

supplemented by a nonlinear term of the form

$$\dots + \frac{1 - \sigma^2}{\sigma c_0} (|\mathbf{v}| + 0.3|\mathbf{v}_0|). \quad (33)$$

These forms are totally useless for the present purposes, because  $\sqrt{\omega}$  and  $|\mathbf{v}|$  cannot be extended to functions that obey both causality and reality conditions.

If any, only the basic Helmholtz resonator liner (example 4.1.3) was found to be useful (see below).

## 6 Direct numerical approaches

In the literature some direct numerical approaches are presented. We will discuss here three.

## 6.1 3-parameter model

In [10] Tam & Auriat explored the possibilities of a simple 3-parameter model, which is essentially the mass-spring-damper model discussed in subsection 4.1.1. It was applied in (for example) [11, 12]. For a positive reactance they made the choice of  $K = 0$  (in our notation), and expressed  $p$  in  $v$  and  $v'$ . For a negative reactance they took  $m = 0$ , used the split form (18), and expressed  $p'$  in  $v$  and  $v'$ . The causality condition of the impedance  $Z$  was correctly associated with the location of the pole at  $\omega = 0$ , but the causality condition of the admittance  $Z^{-1}$  was called “stability” condition, which is confusing.

## 6.2 Rational functions in $\omega$

An, at first sight general and universal, approach is a model in the form of a rational function in  $\omega$  (a further generalisation of the model discussed in subsection 4.1.2)

$$Z(\omega) = Z_\infty \frac{\prod_{n=1}^N (\omega - \alpha_n)}{\prod_{n=1}^M (\omega - \beta_n)} = Z_\infty \frac{\sum_{n=1}^N a_n \omega^n}{\sum_{n=1}^M b_n \omega^n} \quad (34)$$

In [13, 14] and [15] this form was used for  $N = 4$  and  $M = 3$ , in [16] with  $N = M = 2$ , and in [17] with  $N = M = 2$  and  $N = M = 4$ . Care is required that the zeros and poles are located in the correct part of the complex plane (*i.e.*, the upper half for the present  $i\omega t$  notation), while at the same time the reality and passivity conditions are fulfilled. In practice this means that  $Z_\infty$  is real and  $\alpha_n$  (and similarly  $\beta_n$ ) are either real or come in pairs such that  $\alpha_n^* = -\alpha_{n'}$  (see figure 3).



Figure 3: Sketch of location of zeros  $\alpha_n$  and poles  $\beta_n$  of  $Z(\omega)$ .

These extra conditions make it practically difficult to use for  $Z$  a Padé approximation, *i.e.*, the best rational-function fit through a given set of data points. This Padé approximation is unique, but there is no guarantee that the result satisfies the fundamental conditions of causality etc. Therefore, it is not appropriate to call this form a Padé expression, like in [16].

The resulting time domain form is either:  $p$  expressed in  $v, v', \dots, v^{(M-N)}$ , and an integral over  $v$ , or: a relation between  $p, p', \dots, p^{(N)}$  and  $v, v', \dots, v^{(M)}$ . The occurrence of the derivatives makes this method less attractive in numerical applications.

## 6.3 The $z$ -transform, or a rational function in $e^{-i\omega\kappa}$

In order to avoid the need of evaluating derivatives, Ozyoruk & Long proposed a model where the frequency dependence is always through the group  $e^{-i\omega\kappa}$ . In the jargon of computational electromagnetics, this is called: the  $z$ -transform. By taking  $\kappa$  equal to a multiple of the numerical timestep no other data than already available is needed for an exact representation. In our formulation, the idea is as follows.



Consider the following function  $F(z)$  in the complex variable  $z$ , analytic on the unit disk  $|z| < 1$ . Although not absolutely necessary<sup>2</sup>, it is convenient to assume that  $F$  is a rational function. At least in the unit disk,  $F$  has a Taylor series expansion.

$$F(z) = F_\infty \frac{\prod_{n=1}^N (z - r_n)}{\prod_{n=1}^M (z - s_n)} = \sum_{n=0}^{\infty} a_n z^n. \quad (35)$$

Select a real number  $\kappa$ , and define the impedance function by

$$Z(\omega) = F(e^{-i\omega\kappa}). \quad (36)$$

If  $|r_n|, |s_n| \geq 1$ ,  $F(z)$  is analytic and non-zero on the unit disc  $|z| < 1$ , with the result that  $Z$  is analytic and non-zero in  $\text{Im}(\omega) < 0$ . If  $F_\infty$  is real, and  $F$ 's poles and zeros are either real or come in pairs, such that  $r_n^* = r_{n'}$  and  $s_n^* = s_{n'}$ ,  $Z$  satisfies the reality condition (see figure 4). Automatically all  $a_n$  will be real. If we are able to make sure that  $\text{Re}(F) \geq 0$  along the unit circle  $|z| = 1$  (this is more difficult), the impedance is passive.

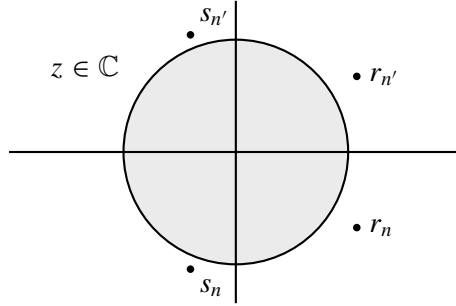


Figure 4: Sketch of location of zeros  $r_n$  and poles  $s_n$  of  $F(z)$ .

The inverse Fourier transform of  $Z$  is now the  $\delta$ -comb

$$z(t) = 2\pi \sum_{n=0}^{\infty} a_n \delta(t - n\kappa) \quad (37)$$

which follows from the relation

$$\delta(t - n\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega n\kappa} e^{i\omega t} d\omega. \quad (38)$$

The key idea is now to take  $\kappa = \Delta t$ , or a multiple, to synchronize the  $\delta$ -comb with the numerical grid of stepsize  $\Delta t$ . We find then in time domain

$$p(t) = \sum_{n=0}^{\infty} a_n v(t - n\Delta t). \quad (39)$$

Note that the infinite series converges exponentially if the poles and zeros are taken outside (not on) the unit disk. The larger  $|r_n| - 1$  and  $|s_n| - 1$  are taken, the faster the series will converge.

<sup>2</sup>To construct such a function in general is related to the Riemann-Hilbert problem [18]

If the infinite series is inconvenient, we can expand the products and write

$$F(z) = \frac{\sum_{n=1}^N c_n z^n}{\sum_{n=1}^M d_n z^n} \quad (40)$$

to get

$$\sum_{n=1}^M d_n p(t - n \Delta t) = \sum_{n=1}^N c_n v(t - n \Delta t). \quad (41)$$

Note that the basic Helmholtz resonator model is almost of the present form. If  $m = 0$  and we identify  $2L/c_0 = \Delta t$  we obtain from (28)

$$Z(\omega) = \frac{R(1 - e^{-i\omega\Delta t}) + 1 + e^{-i\omega\Delta t}}{1 - e^{-i\omega\Delta t}} = \frac{(R + 1) - (R - 1)e^{-i\omega\Delta t}}{1 - e^{-i\omega\Delta t}} \quad (42)$$

The zero in the  $z$ -plane is found at  $\frac{R+1}{R-1}$ , which is indeed less than  $-1$  for  $0 < R < 1$  and greater than  $1$  for  $R > 1$ . The pole is at  $z = 1$ .

The absence of derivatives makes the present method very desirable, especially if we are able to define a parameterised family of impedance models satisfy the fundamental conditions automatically. In the next section we will propose such a family, based on the Helmholtz model of above.

## 7 Extended Helmholtz Resonator model

Inspired by the Helmholtz resonator model and the  $z$ -transform method we will now propose an impedance model that is rich enough to represent any impedance at any given design frequency, needs no additional time derivatives in its time-domain representation and is guaranteed physical, *i.e.*, satisfies the fundamental conditions. We will call this the *Extended Helmholtz Resonator Model*.

### 7.1 Frequency domain

In comparison with the basic Helmholtz resonator of subsection 4.1.3 we have applied the following changes.

1. Damping  $\varepsilon > 0$  in the cavity's fluid;
2. A varying cavity reactance by parameter  $\beta > 0$ ;
3. Cavity depth  $\sim$  multiple of time step:  $2L/c_0 = v\Delta t$ ;

resulting into

$$Z(\omega) = R + i\omega m - i\beta \cot(\frac{1}{2}\omega v\Delta t - i\frac{1}{2}\varepsilon). \quad (43)$$

See figure 5 for an example of the model as a function of  $\omega$ . It is straightforward to check that the model satisfies the reality condition. The model is passive:

$$0 < \text{Re}(Z) = R + \beta \frac{\sinh(\varepsilon)}{\cosh(\varepsilon) - \cos(\omega v\Delta t)} \in [R + \beta \tanh(\frac{1}{2}\varepsilon), R + \beta \coth(\frac{1}{2}\varepsilon)]. \quad (44)$$

The model is causal, since the poles are located at

$$\omega = \frac{1}{v\Delta t}(2n\pi + i\varepsilon), \quad n \in \mathbb{Z}, \quad (45)$$

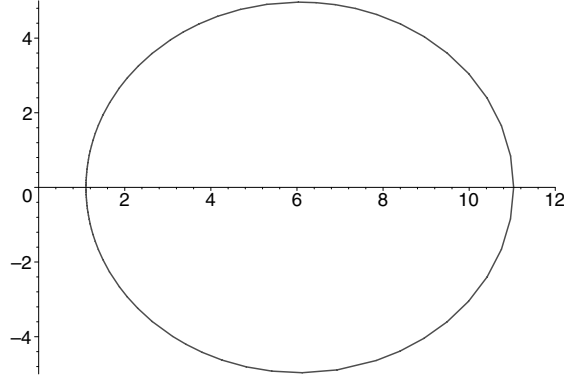


Figure 5: Extended Helmholtz resonator impedance as a function of frequency.  $R = 1$ ,  $m = 0$ ,  $\beta = 1$ ,  $\varepsilon = 0.2$ ,  $\Delta t = 1$ ,  $v = 2$ ,  $\omega = 0 \dots \pi$ .

*i.e.*, in the upper complex half plane, while for  $\omega = \xi - i\eta$  we have

$$\operatorname{Re}(Z) = R + m\eta + \beta \frac{\sinh(\eta v \Delta t + \varepsilon)}{\cosh(\eta v \Delta t + \varepsilon) - \cos(\xi v \Delta t)} > 0 \quad (46)$$

for any  $\eta > 0$ . So for  $\operatorname{Im}(\omega) < \varepsilon/v\Delta t$  we have

$$Z(\omega) = R + i\omega m + \beta + 2\beta \sum_{n=1}^{\infty} e^{-i\omega n v \Delta t - \varepsilon n}, \quad (47)$$

leading to

$$\frac{z(t)}{2\pi} = R\delta(t) + m\delta'(t) + \beta\delta(t) + 2\beta \sum_{n=1}^{\infty} e^{-\varepsilon n} \delta(t - n v \Delta t). \quad (48)$$

## 7.2 Constructing an impedance that satisfies a design condition

Finally we have to fix the parameters for a given design impedance  $Z = Z_0 = R_0 + iX_0$  at a frequency  $\omega = \omega_0 > 0$ . So we have to find parameter values that satisfy

$$R_0 = R + \beta \frac{\sinh(\varepsilon)}{\cosh(\varepsilon) - \cos(\omega_0 v \Delta t)} \quad (49a)$$

$$X_0 = \omega_0 m - \beta \frac{\sin(\omega_0 v \Delta t)}{\cosh(\varepsilon) - \cos(\omega_0 v \Delta t)} \quad (49b)$$

Suppose that a wave period  $T = 2\pi/\omega_0$  is divided in  $N$  parts (typically  $N = 20$ ), *i.e.*,  $T = N\Delta t$ . So

$$\omega_0 v \Delta t = \frac{2\pi v}{N}. \quad (50)$$

If  $X_0 = 0$ , we are ready by taking  $m = \beta = 0$ , so in the following we will assume that  $X_0 \neq 0$ . From a numerical-algorithmic point of view, a most convenient choice is

$$m = 0 \quad (51)$$

because it relieves us of the task to calculate  $v'(t)$  without flow, and no second derivative with mean flow. In that case we have no other way to comply with the sign of  $X_0$  than by selecting  $\sin(\omega_0\nu\Delta t)$  opposite in sign of  $X_0$ :

$$\text{if } X_0 < 0, \text{ we need to take } \nu \in \{1, \dots, \lceil \frac{1}{2}N \rceil - 1\}, \quad (52a)$$

$$\text{if } X_0 > 0, \text{ we need to take } \nu \in \{\lfloor \frac{1}{2}N \rfloor + 1, \dots, N - 1\}. \quad (52b)$$

If we now choose  $\varepsilon$  between the following limits

$$0 \leq \varepsilon \leq \operatorname{arsinh} \left( -\frac{R_0}{X_0} \sin(\omega_0\nu\Delta t) \right), \quad (53)$$

then  $\beta$  is determined from  $X_0$  and  $R$  from  $R_0$  as follows.

$$\beta = -X_0 \frac{\cosh(\varepsilon) - \cos(\omega_0\nu\Delta t)}{\sin(\omega_0\nu\Delta t)} \quad (54a)$$

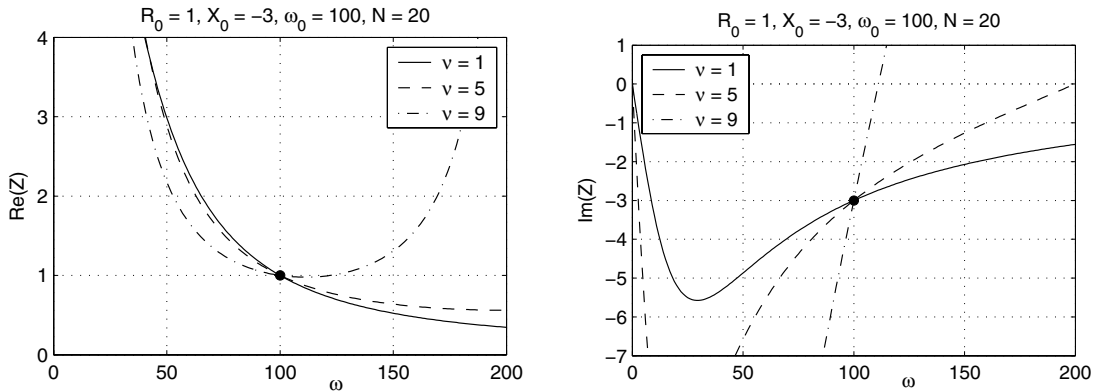
$$R = R_0 + X_0 \frac{\sinh(\varepsilon)}{\sin(\omega_0\nu\Delta t)}. \quad (54b)$$

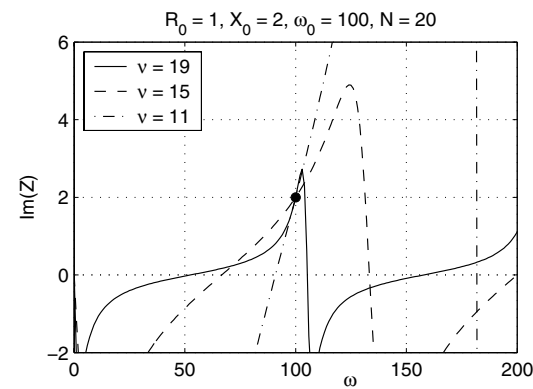
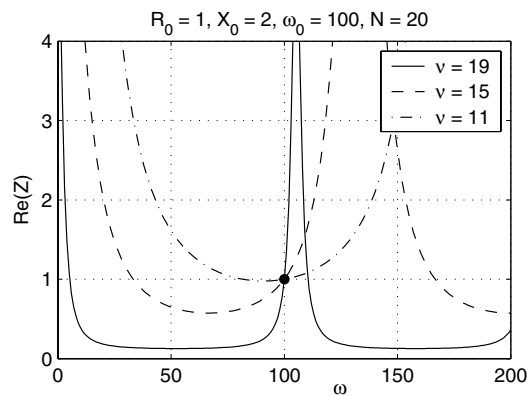
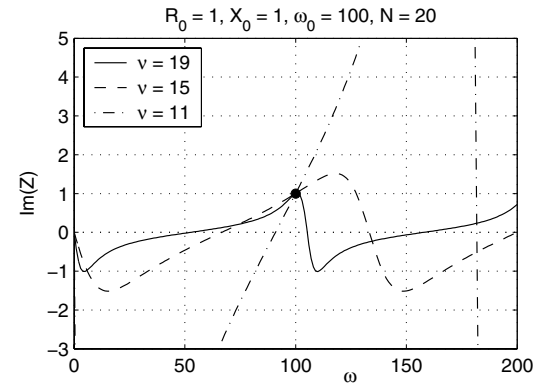
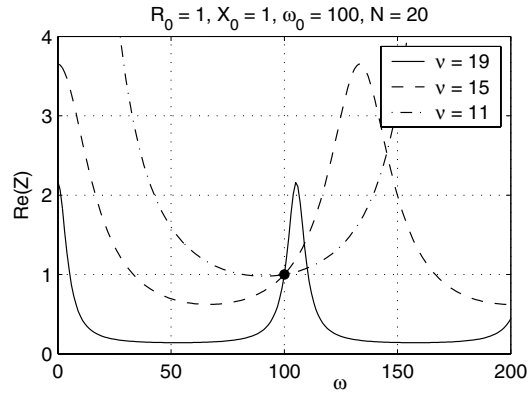
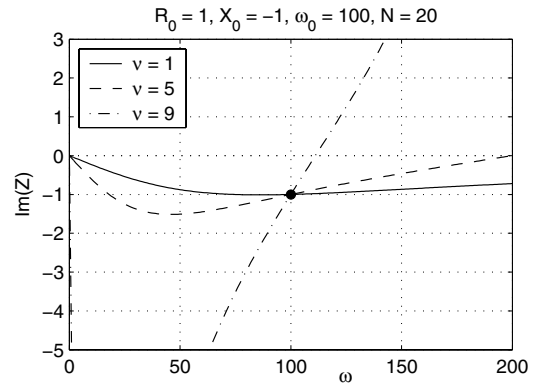
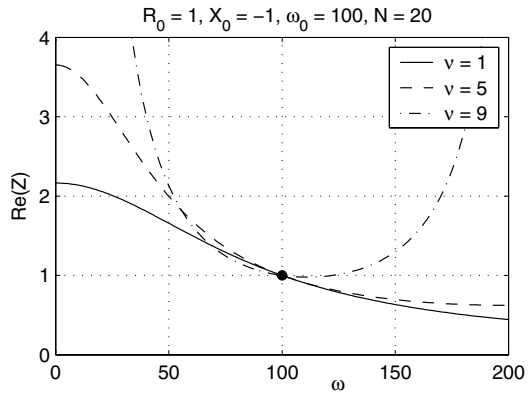
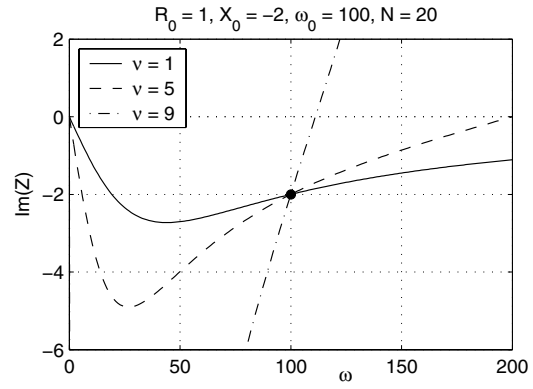
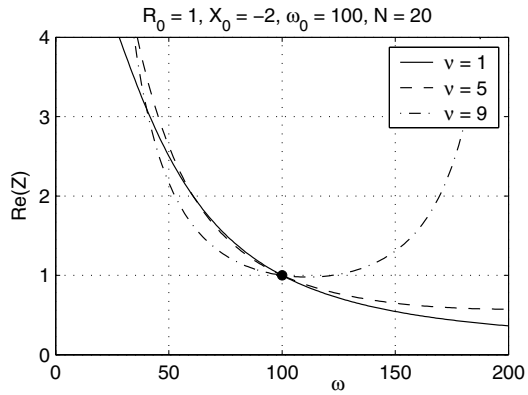
The restriction on  $\varepsilon$  is to make sure that  $R \geq 0$ .

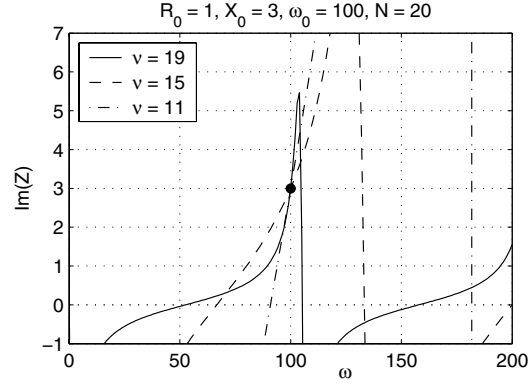
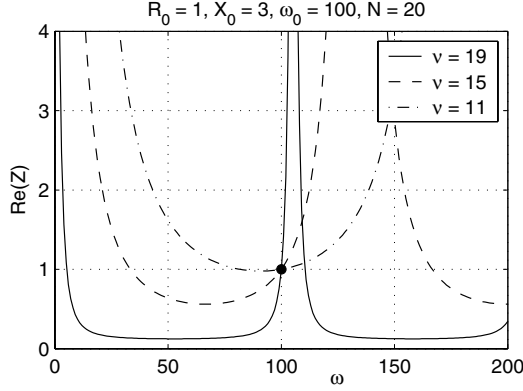
### 7.3 Parameter study

A smooth behaviour of  $Z$  in  $\omega \in \mathbb{R}$  is obtained by pushing the poles (45) away from the real axis, by choosing  $\varepsilon$  as large as is reasonable, for example  $\varepsilon = 0.9 \operatorname{arsinh}(-R_0/X_0 \sin(\omega_0\nu\Delta t))$ .

In order to investigate the effect of the choice of parameter  $\nu$  on the smoothness and variability of  $Z$ , we made a series of plots of  $\operatorname{Re}(Z)$  and  $\operatorname{Im}(Z)$ , as a function of  $\omega$ , for various  $\nu$  and  $Z_0$ . In all cases we selected  $\varepsilon = 0.9 \operatorname{arsinh}(-R_0/X_0 \sin(\omega_0\nu\Delta t))$ . We took  $R_0 = 1$ ,  $\omega_0 = 100$  and  $N = 20$  fixed, and varied  $X_0$  and  $\nu$  as:  $X_0 = -3, -2, -1, 1, 2, 3$  and  $\nu = 1, 5, 9$  for  $X_0 < 0$ , and  $\nu = 19, 15, 11$  for  $X_0 > 0$ . The results for  $0 \leq \omega \leq 2\omega_0$  are shown below.







The conclusion that may be drawn (finer details depend on the other parameters), is that for  $X_0 < 0$  the best choice, *i.e.*, where  $Z$  varies not too much near  $Z_0$ , is for  $\nu = 1$ . For  $X_0 > 0$  on the other hand, it is better to be halfway between  $\nu = \frac{1}{2}N$  and  $N$ .

## 7.4 Time domain

In time domain we have thus

$$p(t) = (R + \beta)v(t) + 2\beta \sum_{n=1}^{\infty} e^{-\varepsilon n} v(t - n\nu\Delta t). \quad (55)$$

or

$$p(t) - e^{-\varepsilon} p(t - \nu\Delta t) = (R + \beta)v(t) - (R - \beta)e^{-\varepsilon} v(t - \nu\Delta t). \quad (56)$$

It is not immediately clear which form is preferred in view of the manipulations of  $p$  that are needed with mean flow.

Without flow we could have had  $m \neq 0$ . With flow, however, we have already a derivative, so here it is better avoided. We have

$$p'(t) + \mathbf{v}_0 \cdot \nabla p(t) + \mathbf{v}_0 \cdot (\mathbf{n} \cdot \nabla \mathbf{n}) p(t) = (R + \beta)v'(t) + 2\beta \sum_{n=1}^{\infty} e^{-\varepsilon n} v'(t - n\nu\Delta t). \quad (57)$$

or

$$\begin{aligned} [p' + \mathbf{v}_0 \cdot \nabla p + \mathbf{v}_0 \cdot (\mathbf{n} \cdot \nabla \mathbf{n}) p]_t - e^{-\varepsilon} [p' + \mathbf{v}_0 \cdot \nabla p + \mathbf{v}_0 \cdot (\mathbf{n} \cdot \nabla \mathbf{n}) p]_{t-\nu\Delta t} \\ = (R + \beta)v'(t) - (R - \beta)e^{-\varepsilon} v'(t - \nu\Delta t). \end{aligned} \quad (58)$$

## 8 An exact solution

In order to understand the behaviour of an impedance in time-domain, we will investigate a concrete reflection problem with the extended Helmholtz resonator impedance type, introduced in section 7. The following analysis generalises [5].

Consider the 1D acoustic wave equations (in dimensional form) for velocity perturbations  $v$ , perturbations pressure  $p$ , and density perturbations  $\rho$

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial v}{\partial x} = 0, \quad \rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0, \quad p = c_0^2 \rho. \quad (59)$$

Consider an arbitrary plane wave  $p_i = f(t - x/c_0)$  incident from  $x < 0$ , and reflecting into  $p_r = g(t + x/c_0)$  by an impedance wall at  $x = 0$ , with impedance  $Z(\omega)$  given by

$$Z(\omega) = \rho_0 c_0 \left[ R + i\omega m - i \cot(\omega L/c_0 - i\frac{1}{2}\varepsilon) \right]. \quad (60)$$

The total acoustic field is given for  $x < 0$  by:

$$p(x, t) = f(t - x/c_0) + g(t + x/c_0), \quad (61a)$$

$$v(x, t) = \frac{1}{\rho_0 c_0} \left( f(t - x/c_0) - g(t + x/c_0) \right). \quad (61b)$$

The reflected wave  $g$  is determined via the impedance condition.

$$f(t) + g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(t - \tau) (f(\tau) - g(\tau)) d\tau. \quad (62)$$

where

$$\frac{z(t)}{2\pi} = \rho_0 c_0 \left[ R\delta(t) + m\delta'(t) + \delta(t) + 2 \sum_{n=1}^{\infty} e^{-\varepsilon n} \delta(t - 2nL/c_0) \right]. \quad (63)$$

For any incident wave starting at some finite time ( $t = 0$ ) we have  $f(t) = 0$  for  $t < 0$ , so that all in all the infinite integral reduces to an integration over the interval  $[0, t]$ . For any time  $t$ ,  $g(t)$  is built up from  $f(t)$  and the history of  $f$  and  $g$  along  $[0, t]$ .

## 8.1 Solution

When we absorb  $\rho_0 c_0$  into the definition of  $z$ , we have

$$(R + 2)g(t) + mg'(t) = Rf(t) + mf'(t) + 2 \sum_{n=1}^{\infty} e^{-\varepsilon n} (f(t_n) - g(t_n)), \quad (64)$$

where  $t_n = t - 2nL/c_0$ . If  $m \neq 0$ , this differential equation in  $g$  has solution

$$g(t) = f(t) - \frac{2}{m} \int_{-\infty}^t \exp\left(-\frac{R+2}{m}(t-\tau)\right) \left[ f(\tau) - \sum_{n=1}^{\infty} e^{-\varepsilon n} (f(t_n) - g(t_n)) \right] d\tau. \quad (65)$$

If  $m = 0$  we have

$$g(t) = \frac{R}{R+2} f(t) + \frac{2}{R+2} \sum_{n=1}^{\infty} e^{-\varepsilon n} (f(t_n) - g(t_n)). \quad (66)$$

Note that the integral really starts from whenever  $f \neq 0$ , say, from  $\tau = 0$ , which means that the summation is finite and ends at

$$N(t) = \lfloor tc_0/2L \rfloor. \quad (67)$$

If  $f(t) = \delta(t)$  we can evaluate the integral more explicitly to get ( $m \neq 0$ )

$$g(t) = \delta(t) - \frac{2}{m} \sum_{n=0}^{N(t)} \exp\left(-\frac{R+2}{m}t_n - \varepsilon n\right) L_n^{(-1)}(2t_n/m) \quad (68)$$

and ( $m = 0$ )

$$g(t) = \frac{R}{R+2} \delta(t) + \frac{4}{R(R+2)} \sum_{n=1}^{N(t)} \left( \frac{R}{R+2} \right)^n e^{-\varepsilon n} \delta(t_n). \quad (69)$$

where  $L_n^{(-1)}$  is a generalised Laguerre polynomial [9, 19]:

$$L_n^{(-1)}(x) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n-1}{k-1} x^k \quad (70)$$

where  $L_0^{(-1)}(x) = 1$ ,  $L_1^{(-1)}(x) = -x$ , etc. It may be noted that  $L_n^{(-1)}(x)$  has the generating function

$$\begin{aligned} \mathcal{G}(x, z) = \exp\left(-\frac{xz}{1-z}\right) &= 1 - xz + (-x + \frac{1}{2}x^2)z^2 + (-x + x^2 - \frac{1}{6}x^3)z^3 \\ &+ (-x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4)z^4 + (-x + 2x^2 - x^3 + \frac{1}{6}x^4 - \frac{1}{120}x^5)z^5 \\ &+ (-x + \frac{5}{2}x^2 - \frac{5}{3}x^3 + \frac{5}{12}x^4 - \frac{1}{24}x^5 + \frac{1}{720}x^6)z^6 + \mathcal{O}(z^7) \end{aligned} \quad (71)$$

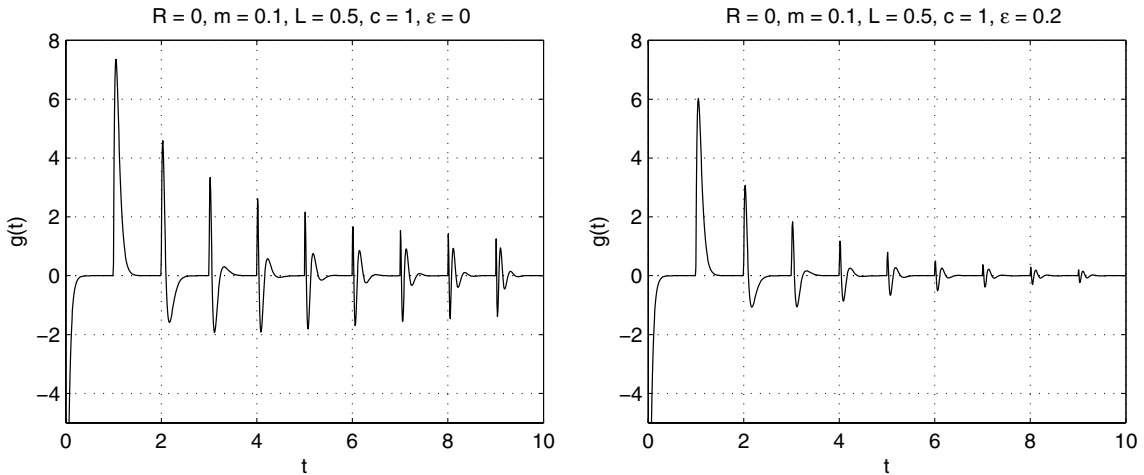
From this solution (68, 69) we can solve for any initial profile by writing

$$f(t) = f_0(t) = \int_{-\infty}^{\infty} f_0(t - \tau) \delta(\tau) d\tau \quad (72)$$

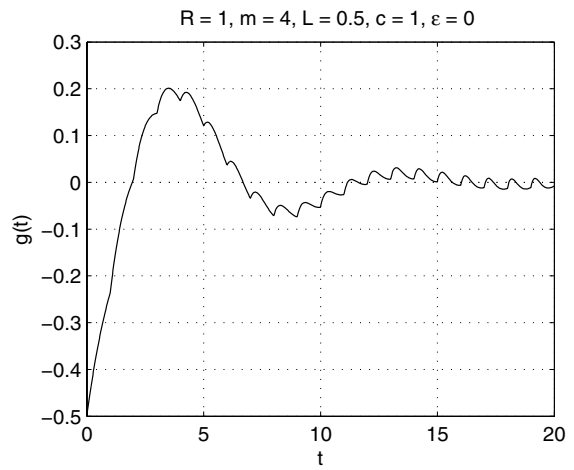
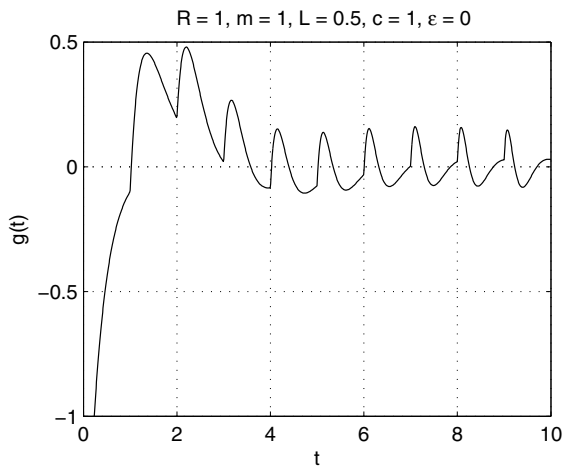
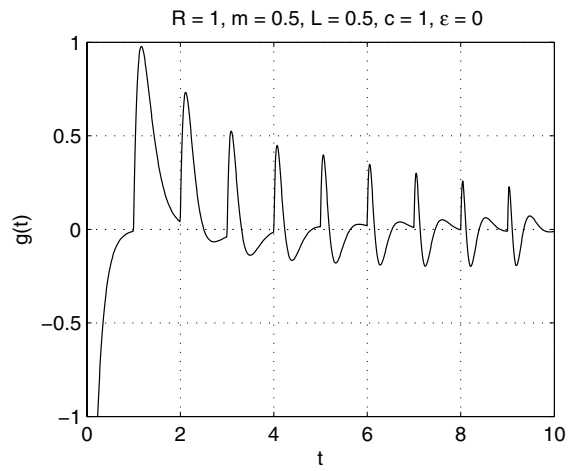
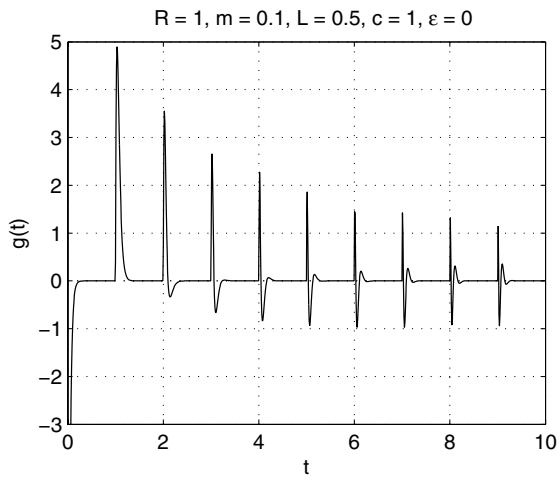
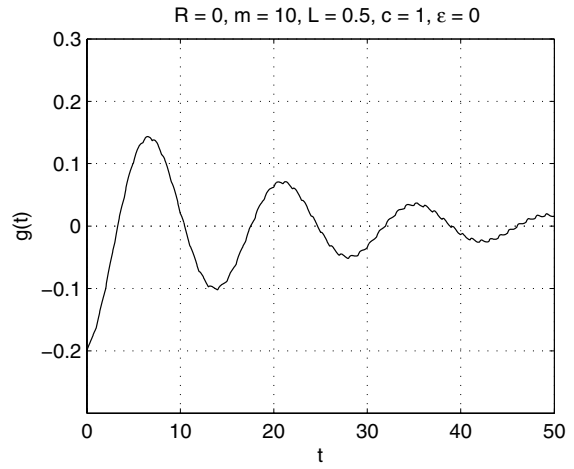
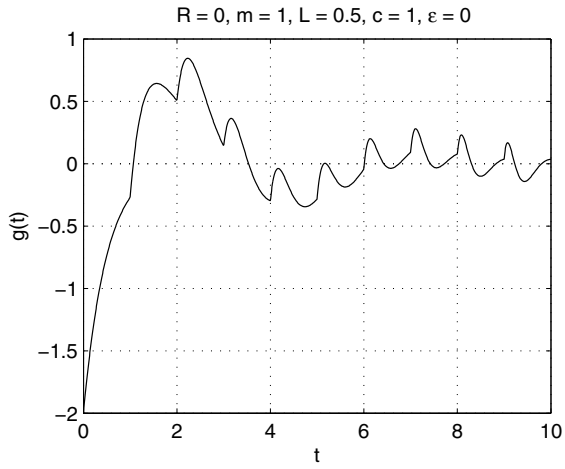
Assuming that  $f_0(t) \equiv 0$  for  $t < 0$ , we have for  $m = 0$

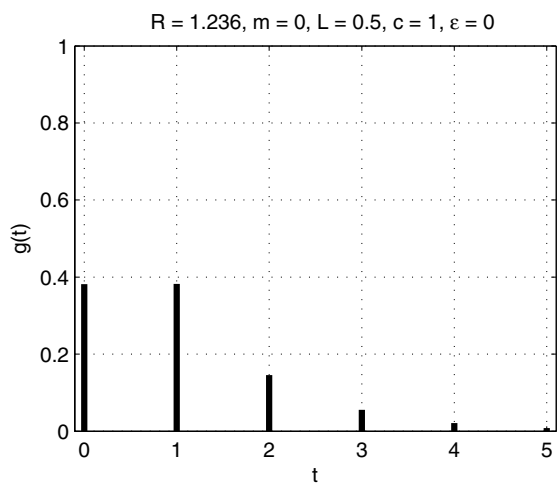
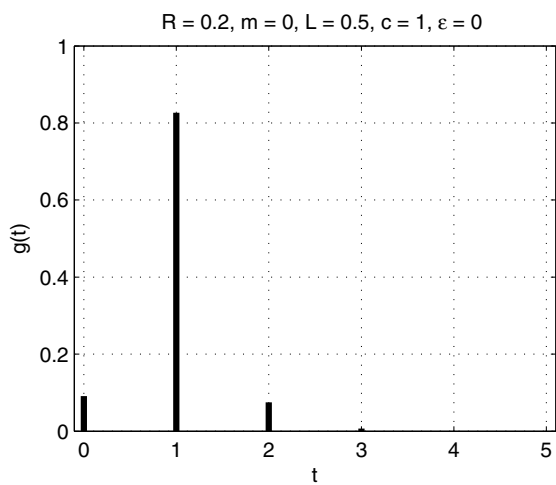
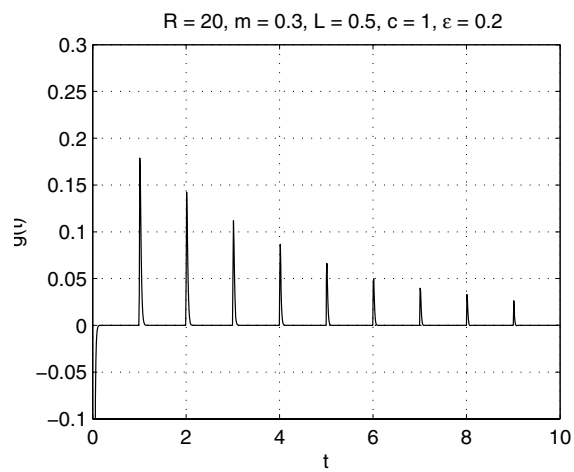
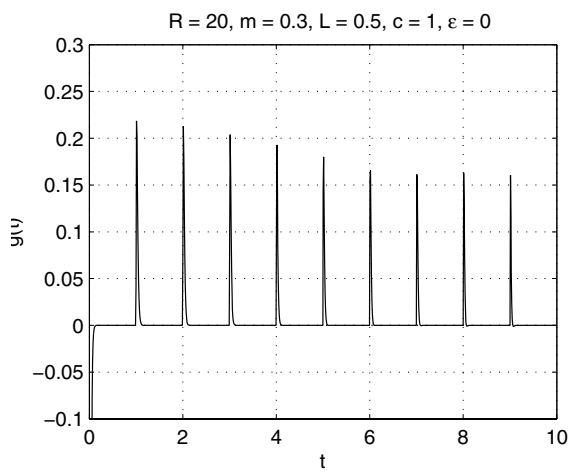
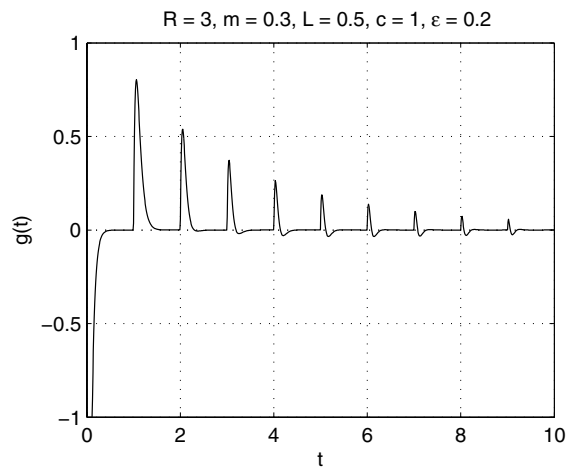
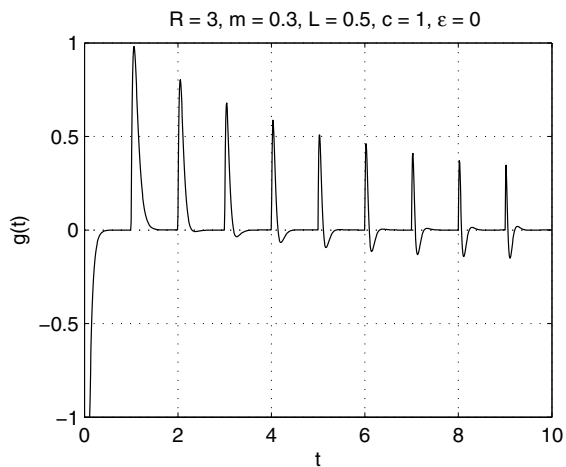
$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} f_0(t - \tau) \left[ \frac{R}{R+2} \delta(\tau) + \frac{4}{R(R+2)} \sum_{n=1}^{N(\tau)} \left( \frac{R}{R+2} \right)^n e^{-2\varepsilon n} \delta(\tau_n) \right] d\tau \\ &= \frac{R}{R+2} f_0(t) + \frac{4}{R(R+2)} \sum_{n=1}^{N(t)} \left( \frac{R}{R+2} \right)^n e^{-2\varepsilon n} f_0(t_n). \end{aligned} \quad (73)$$

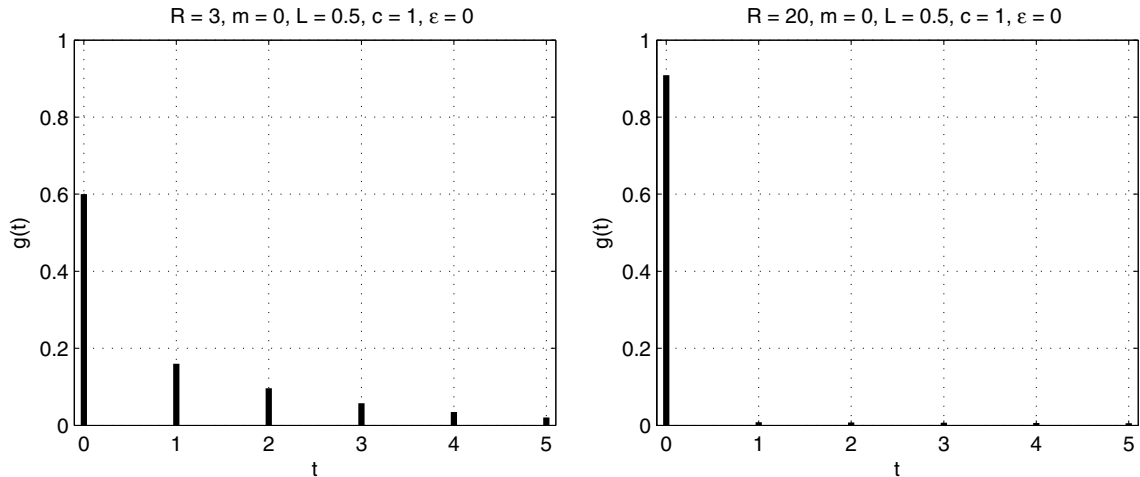
In the following we present some graphical examples of this solution  $g(t)$ , *i.e.*, (68) and (69). When  $m \neq 0$  the first  $\delta$ -function is not drawn. When  $m = 0$ , the  $\delta$ -functions are drawn like a solid line of a length, proportional to its amplitude.











## 9 Conclusions

The problem of translating an impedance boundary condition to time domain involves the Fourier transform of the impedance function. This requires extending the definition of the impedance not only to all real frequencies but to the whole complex frequency plane. Not any extension, however, is physically possible. The problem should remain causal, the variables real, and the wall passive. This leads to necessary conditions for an impedance function.

Various methods of extending the impedance that are available in the literature are discussed. A most promising one is the so-called  $z$ -transform, which is nothing but an impedance that is functionally dependent on a suitable complex exponent  $e^{-i\omega\kappa}$ . By choosing  $\kappa$  a multiple of the time step of the numerical algorithm, this approach fits very well with the underlying numerics, because the impedance becomes in time domain a delta-comb function and gives thus an exact relation on the grid points.

A recipe is proposed for an impedance which is based on the Helmholtz resonator model. This has the advantage that relatively easily the mentioned necessary conditions can be satisfied in advance. At given frequency, the impedance is made exactly equal to a given design value. Rules of thumb are derived to produce an impedance which varies only moderately in frequency near design condition.

An explicit solution is given of a pulse reflecting in time domain at a Helmholtz resonator impedance wall that provides some insight into the reflection problem in time domain and at the same time may act as an analytical test case for numerical implementations.

## Appendix

### Causality

No physical process can exist for all time. A process  $f(t)$  that starts by some cause at some finite time  $t = t_0$ , while it vanishes before  $t_0$ , is called *causal*. By analytic continuation in  $\omega$  we can show that the corresponding Fourier transform

$$\hat{f}(\omega) = \int_{t_0}^{\infty} f(t) e^{-i\omega t} dt \quad (74)$$

has the property that  $\hat{f}(\omega)$  is analytic in the lower complex half-plane  $\text{Im}(\omega) < 0$ . So this is a necessary condition on  $\hat{f}$  for  $f$  to be causal. A sufficient condition is the following *causality condition* [20].

**Theorem 3 (Causality Condition)** *If  $\hat{f}(\omega)$  is analytic in  $\text{Im}(\omega) \leq 0$ ,  $|\hat{f}(\omega)|^2$  is integrable along the real axis, and there is a real  $t_0$  such that  $\hat{f}(\omega) e^{i\omega t_0} \rightarrow 0$  uniformly with regard to  $\arg(\omega)$  for  $|\omega| \rightarrow \infty$  in the lower complex half plane, then  $f(t)$  is causal, and vanishes for  $t < t_0$ .*

**Proof.** It is no restriction for the proof to assume  $t_0 = 0$ . Consider, for  $t < 0$ , the integral  $\int \hat{f}(\omega) e^{i\omega t} d\omega$  along the real contour  $[-R, R]$  closed via a semi-circle of radius  $R$  in the lower complex half plane. As the integrand is analytic the integral is zero. Let  $R \rightarrow \infty$ . The contribution  $I_R$  of the integral along the semi-circle tends to zero, because

$$|I_R| \leq \int_0^\pi |\hat{f}(\omega)| e^{-|t|R \sin \theta} R d\theta \leq 2R \max_\theta |\hat{f}(\omega)| \int_0^{\frac{1}{2}\pi} e^{-|t|R 2\theta/\pi} d\theta \rightarrow 0, \quad (75)$$

where  $\omega = R e^{-i\theta}$ . So the contribution from the real axis, being equal to  $2\pi f(t)$ , is also zero.  $\square$

#### Notes.

- The *lower* complex half-space becomes the *upper* half-space if the opposite Fourier sign convention is taken.
- The theorem is also applicable to non-convergent Fourier integrals that can be interpreted as generalised functions, provided a suitable limit is taken.

#### Example 1.

The Fourier transform  $\hat{f}(\omega) = (p + i\omega)^{-1}$  is causal if  $p > 0$ , as may be confirmed by the inverse transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{p + i\omega} d\omega = H(t) e^{-pt} = \begin{cases} e^{-pt} & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases} \quad (76)$$

In the limit of no damping ( $p \downarrow 0$ ) the singularity at  $\omega = ip$  moves to  $\omega = 0$  on the real axis. The integral is to be interpreted via a suitable indentation of the contour *under* the pole in order to retain causality.  $\square$

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