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Thin Layer Flow Along Arbitrary Curved Surfaces

To model the inertia and gravity driven stationary flow of a thin layer of water along a curved surface (for example, in a washing bowl, a toilet, or other technical applications) the equations for free-surface potential flow (Laplace, Bernoulli, b.c.) are rewritten in surface-bound, curvilinear orthogonal coordinates. Assuming a small parameter, measuring the thin fluid layer, a systematic perturbation analysis is made, producing, to leading order, equations similar to the eikonal and energy equation in ray theory. These hyperbolic equations can be integrated along streamlines, with explicit results for some geometries. In these equations the smoothing backreaction of the pressure is decoupled, leading to the possibility of singular lines, being the envelope of crossing streamlines (caustics).

1. The model

Consider along a surface $S(x,y,z)=0$, with gravity field $-g\vec{\mathbf{e}}_z$, a thin layer of incompressible, inviscid, irrotational, stationary flow (water), at constant atmospheric pressure at the free surface $S^*(x,y,z)=0$. Inside the layer we have a potential Φ, a pressure p, and a constant density ρ, satisfying

$$\nabla^2\Phi(x,y,z)=0 \qquad \text{(mass conservation)} \qquad (1)$$

$$p/\rho+\tfrac{1}{2}|\nabla\Phi|^2+gz=\text{constant along a streamline} \qquad \text{(Bernoulli)} \qquad (2)$$

$$\nabla\Phi\cdot\nabla S=0 \ \text{ at } S=0, \quad \text{and} \quad \nabla\Phi\cdot\nabla S^*=0,\ \ p=\text{constant} \ \text{ at } S^*=0 \qquad \text{(boundary conditions)} \qquad (3)$$

2. Local Coordinates

Assume we can define a curvilinear orthogonal coordinate system (σ,τ,ν), given by $x=x(\sigma,\tau,\nu)$, $y=y(\sigma,\tau,\nu)$, $z=z(\sigma,\tau,\nu)$, and attached to the surface such that $\nu=0$ corresponds to $S=0$, where $\nu>0$ is the wet side and $\nu=\varepsilon H(\sigma,\tau)$ is the free surface (ε is small). For simplicity we will write $\Phi(x,y,z)=\Phi(\sigma,\tau,\nu)$. We utilize the calculus of vectors and tensors in curvilinear orthogonal coordinates [1] and introduce

$$\text{scale factors: } h_\alpha^2=\sum_\xi\Big(\frac{\partial\xi}{\partial\alpha}\Big)^2; \qquad \alpha=\sigma,\tau,\nu; \quad \xi=x,y,z$$

$$\text{unit vectors: } \vec{\mathbf{e}}_\alpha=\frac{1}{h_\alpha}\sum_\xi\Big(\frac{\partial\xi}{\partial\alpha}\Big)\vec{\mathbf{e}}_\xi, \qquad \frac{\vec{\mathbf{e}}_\beta}{h_\beta}\cdot\frac{\partial\vec{\mathbf{e}}_\alpha}{\partial\beta}=\sum_\xi\Bigg(\frac{1}{h_\beta^2}\Big(\frac{\partial\xi}{\partial\beta}\Big)\frac{\partial}{\partial\beta}\Big(\frac{1}{h_\alpha}\frac{\partial\xi}{\partial\alpha}\Big)\Bigg)$$

$$\text{hence: } \quad \nabla=\sum_\alpha\frac{\vec{\mathbf{e}}_\alpha}{h_\alpha}\Big(\frac{\partial}{\partial\alpha}\Big), \qquad \nabla\Phi=\sum_\alpha\frac{\vec{\mathbf{e}}_\alpha}{h_\alpha}\Big(\frac{\partial\Phi}{\partial\alpha}\Big), \qquad |\nabla\Phi|^2=\sum_\alpha\frac{1}{h_\alpha^2}\Big(\frac{\partial\Phi}{\partial\alpha}\Big)^2$$

And so we have for the Laplace equation and boundary conditions

$$\sum_\alpha\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\Big(\frac{1}{h_\alpha}\frac{\partial\Phi}{\partial\alpha}\Big)+\sum_\alpha\Bigg(\frac{1}{h_\alpha}\frac{\partial\Phi}{\partial\alpha}\sum_\beta\Big(\frac{\vec{\mathbf{e}}_\beta}{h_\beta}\cdot\frac{\partial\vec{\mathbf{e}}_\alpha}{\partial\beta}\Big)\Bigg)=0 \qquad (4)$$

$$\frac{\partial\Phi}{\partial\nu}=0 \text{ at } \nu=0, \ \frac{1}{h_\nu^2}\frac{\partial\Phi}{\partial\nu}-\varepsilon\frac{1}{h_\sigma^2}\frac{\partial H}{\partial\sigma}\frac{\partial\Phi}{\partial\sigma}-\varepsilon\frac{1}{h_\tau^2}\frac{\partial H}{\partial\tau}\frac{\partial\Phi}{\partial\tau}=0, \ \sum_\alpha\frac{1}{h_\alpha^2}\Big(\frac{\partial\Phi}{\partial\alpha}\Big)^2+2gz=\text{cnst. at } \nu=\varepsilon H. \qquad (5)$$

3. Perturbation Analysis

For small ε we scale $\nu=\varepsilon\eta$, and expand $\Phi(\sigma,\tau,\eta;\varepsilon)=\Phi_0(\sigma,\tau,\eta)+O(\varepsilon)$, $h_\alpha(\sigma,\tau,\varepsilon\eta)=h_\alpha(\sigma,\tau,0)+O(\varepsilon)$, and $\vec{\mathbf{e}}_\alpha(\sigma,\tau,\varepsilon\eta)=\vec{\mathbf{e}}_\alpha(\sigma,\tau,0)+O(\varepsilon)$, assuming that the surface is sufficiently smooth such that $h_\alpha=O(1)$ and $\frac{\partial}{\partial\alpha}(\vec{\mathbf{e}}_\alpha/h_\alpha)=O(1)$. We substitute this in the equations and boundary conditions, collect like powers of ε, and obtain that $\Phi_0=\phi(\sigma,\tau)$ is a function independent of η, satisfying

$$\frac{1}{h_\sigma^2}\Big(\frac{\partial\phi}{\partial\sigma}\Big)^2+\frac{1}{h_\tau^2}\Big(\frac{\partial\phi}{\partial\tau}\Big)^2+2gz(\sigma,\tau,0)=\text{constant}, \qquad (6)$$

$$\frac{1}{h_\sigma}\frac{\partial}{\partial\sigma}\Big(\frac{H}{h_\sigma}\frac{\partial\phi}{\partial\sigma}\Big)+\Big(\frac{H}{h_\sigma}\frac{\partial\phi}{\partial\sigma}\Big)\sum_\alpha\Big(\frac{\vec{\mathbf{e}}_\alpha}{h_\alpha}\cdot\frac{\partial\vec{\mathbf{e}}_\sigma}{\partial\alpha}\Big)+\frac{1}{h_\tau}\frac{\partial}{\partial\tau}\Big(\frac{H}{h_\tau}\frac{\partial\phi}{\partial\tau}\Big)+\Big(\frac{H}{h_\tau}\frac{\partial\phi}{\partial\tau}\Big)\sum_\alpha\Big(\frac{\vec{\mathbf{e}}_\alpha}{h_\alpha}\cdot\frac{\partial\vec{\mathbf{e}}_\tau}{\partial\alpha}\Big)=0. \qquad (7)$$

These two equations (6,7) are are very similar to the *eikonal* and *energy* equation of ray theory. Equation (6) is of first order hyperbolic type, and may be solved by integration along characteristics (= streamlines). Define $q_\sigma = \frac{\partial}{\partial\sigma}\phi$, $q_\tau = \frac{\partial}{\partial\tau}\phi$. Then along the curve $(\sigma(t), \tau(t))$, parameterized by t, we have the system of ordinary differential equations

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{q_\sigma}{h_\sigma^2}, \qquad \frac{\mathrm{d}q_\sigma}{\mathrm{d}t} = \frac{1}{h_\sigma^3}\Big(\frac{\partial h_\sigma}{\partial\sigma}\Big)q_\sigma^2 + \frac{1}{h_\tau^3}\Big(\frac{\partial h_\tau}{\partial\sigma}\Big)q_\tau^2 - g\frac{\partial z}{\partial\sigma} \tag{8,9}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{q_\tau}{h_\tau^2}, \qquad \frac{\mathrm{d}q_\tau}{\mathrm{d}t} = \frac{1}{h_\sigma^3}\Big(\frac{\partial h_\sigma}{\partial\tau}\Big)q_\sigma^2 + \frac{1}{h_\tau^3}\Big(\frac{\partial h_\tau}{\partial\tau}\Big)q_\tau^2 - g\frac{\partial z}{\partial\tau} \tag{10,11}$$

with suitable initial conditions $\sigma(0) = \sigma^0$, $\tau(0) = \tau^0$, $\frac{\partial}{\partial\sigma}\phi(0) = \phi_\sigma^0$, $\frac{\partial}{\partial\tau}\phi(0) = \phi_\tau^0$. Equation (7) is equivalent to $\nabla\cdot(H\nabla\phi) = 0$, so for any given ϕ the height H is found by application of Gauss' theorem, leading to the relation: $H|\nabla\phi|\mathrm{d}\ell$ = constant for any small distance $\mathrm{d}\ell$ between streamlines. The envelopes of the crossing streamlines (caustics) are found by observing that neighbouring streamlines touch each other there [2]. If the streamlines are parametrized by θ (for example, the azimuthal angle of a point source) and formally given by $\Sigma(\sigma, t, \theta) = 0$ and $T(\tau, t, \theta) = 0$, then we have the additional conditions $\frac{\mathrm{d}}{\mathrm{d}\theta}\Sigma(\sigma, t(\theta), \theta) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}\theta}T(\tau, t(\theta), \theta) = 0$, leading to

$$\Sigma_\theta/\Sigma_t = T_\theta/T_t \tag{12}$$

4. Applications

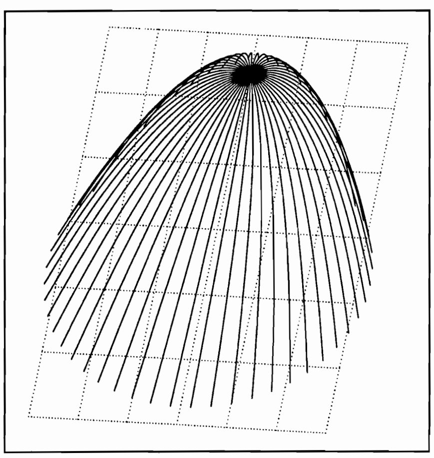
For really arbitrary surfaces the characteristic equations have to be solved numerically. Relatively simple configurations (see the figures) allow also an analytical treatment. For a flat plane, inclined under an angle γ, and described by $x = \sigma\cos\gamma - \nu\sin\gamma$, $y = \tau$, $z = \sigma\sin\gamma + \nu\cos\gamma$, we find the family of streamlines

$$\sigma = \sigma^0 + \phi_\sigma^0 t - \tfrac{1}{2}gt^2\sin\gamma, \qquad \tau = \tau^0 + \phi_\tau^0 t. \tag{13}$$

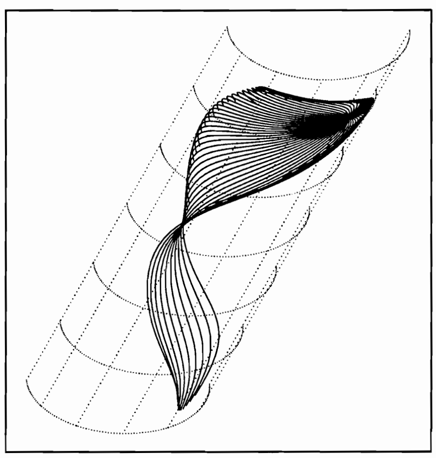
For a point source at the origin, of strength $\phi_\sigma^0 = Q\sin\theta$ and $\phi_\tau^0 = Q\cos\theta$, we have the parabola-shaped caustic

$$\sigma = \frac{g\sin\gamma}{2Q^2}\Big(\frac{Q^4}{g^2\sin^2\gamma} - \tau^2\Big) \tag{14}$$

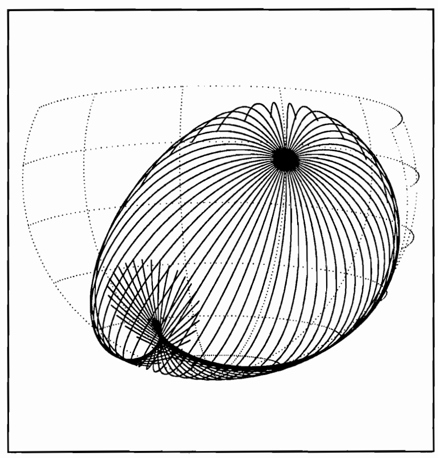
Similar results may be obtained for a flow inside an inclined cylinder, and inside a sphere (with solutions in terms of elliptic integrals). It is instructive to investigate the pattern of crossing streamlines and to compare with experiments.



Flat Plane



Cylinder



Sphere

Acknowledgements

We wish to thank Alistair D. Fitt for his interest.

5. References

1 Batchelor, G.K.: An Introduction to Fluid Mechanics, Cambridge University Press, Cambridge (1967)
2 Whitham, G.B.: Linear and Nonlinear Waves, John Wiley & Sons, New York (1974)

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