An introduction to aeroacoustics

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18 Jul 2004

1 Introduction

Due to the nonlinearity of the governing equations it is very difficult to predict the sound production of fluid flows. This sound production occurs typically at high speed flows, for which nonlinear inertial terms in the equation of motion are much larger than the viscous terms (high Reynolds numbers). As sound production represents only a very minute fraction of the energy in the flow the direct prediction of sound generation is very difficult. This is particularly dramatic in free space and at low subsonic speeds. The fact that the sound field is in some sense a small perturbation of the flow, can, however, be used to obtain approximate solutions.

Aero-acoustics provides such approximations and at the same time a definition of the acoustical field as an extrapolation of an ideal reference flow. The difference between the actual flow and the reference flow is identified as a source of sound. This idea was introduced by Lighthill [68, 69] who called this an *analogy*. A second key idea of Lighthill [69] is the use of integral equations as a formal solution. The sound field is obtained as a convolution of the Green's function and the sound source. The Green's function is the linear response of the reference flow, used to define the acoustical field, to an impulsive point source. A great advantage of this formulation is that random errors in the sound source are averaged out by the integration. As the source also depends on the sound field this expression is not yet a solution of the problem. However, under free field conditions one can often neglect this feedback from the acoustical field to the flow. In that case the integral formulation provides a solution.

When the flow is confined, the acoustical energy can accumulate into resonant modes. Since the acoustical particle displacement velocity can become of the same order of magnitude as the main flow velocity, the feedback from the acoustical field to the sound sources can be very significant. This leads to the occurrence of self-sustained oscillations which we call whistling. In spite of the back-reaction, the ideas of the analogy will appear to remain useful.

As linear acoustics is used to determine a suitable Green's function, it is important to obtain basic insight into properties of elementary solutions of the wave equation. We will focus here on the wave equation describing the propagation of pressure perturbations in a uniform stagnant (quiescent) fluid.

While in acoustics of quiescent media it is rather indifferent whether we consider a wave equation for the pressure or the density we will see that in aero-acoustics the choice of a different variable corresponds to a different choice of the reference flow and hence to another analogy. It seems paradoxical that analogies are not equivalent, since they are all reformulations of the basic equations of fluid dynamics. The reason is that the analogy is used as an approximation. Such an approximation is based on some intuition and usually empirical observations. An example of such an approximation was already quoted above. In free-field conditions we often neglect the influence of the acoustical feedback on the sound sources.

While Lighthill's analogy is very general and useful for order of magnitude estimate, it is less convenient when used to predict sound production by numerical simulations. One of the problems is that the sound source deduced from Lighthill's analogy is spatially rather extended, leading to slowly converging integrals. For low Mach number isothermal flow we will see that aerodynamic sound production is entirely due to mean flow velocity fluctuations, which may be described directly in terms of the underlying vortex dynamics. This is more convenient because vorticity is in general limited to a much smaller region in space than the corresponding velocity field (Lighthill's sound sources). This leads to the idea of using an irrotational flow as reference flow. The result is called Vortex Sound

Theory. Vortex Sound Theory is not only numerically efficient, it also allows us to translate the very efficient vortex-dynamical description of elementary flows directly into sound production properties of these flows.

We present here only a short summary of elements of acoustics and aero-acoustics. The structure of this chapter is inspired by the books of Dowling and Ffowcs Williams [25] and Crighton et. al [16]. A more advanced discussion is provided in text books [102, 132, 82, 35, 6, 16, 49, 47, 48, 119]. The influence of wall vibration is discussed in among others [14, 58, 93].

In the following sections of this chapter we will consider:

- Some fluid dynamics (section 2),
- Free space acoustics (section 3),
- Aero-acoustical analogies (section 4),
- Aero-acoustics of confined flows (section 5),

2 Fluid dynamics

2.1 Mass, momentum and energy equations

We consider the motion of fluids in the continuum approximation. This means that quantities such as the velocity $\boldsymbol{v}$ and the density ρ are smooth functions of space and time coordinates $(\boldsymbol{x}, t)$ [105, 3, 63, 126, 62, 99, 27]. We consider the fundamental equations of mass, momentum and energy applied to an infinitesimally small fluid particle of volume V. We call this a material element. We define the density of the material element equal to ρ, and the mass is therefore simply ρV. As the mass is conserved, i.e.

$$\mathrm{d}(\rho V) = \rho \mathrm{d}V + V \mathrm{d}\rho = 0,$$

the rate of change of the density, observed while moving with the fluid velocity $\boldsymbol{v}$, is equal to minus the dilatation rate:

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\frac{1}{V}\frac{\mathrm{D}V}{\mathrm{D}t} = -\nabla\cdot\boldsymbol{v}$$

where the Lagrangian time derivative $\mathrm{D}\rho/\mathrm{D}t$ is related to the Eulerian time derivative $\partial\rho/\partial t$ by:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + (\boldsymbol{v}\cdot\nabla)\rho. \tag{1}$$

For a cartesian coordinate system $\boldsymbol{x} = (x_1, x_2, x_3)$ we can write this in the index notation:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + v_i\frac{\partial\rho}{\partial x_i} \ \text{ where } \ v_i\frac{\partial\rho}{\partial x_i} = v_1\frac{\partial\rho}{\partial x_1} + v_2\frac{\partial\rho}{\partial x_2} + v_3\frac{\partial\rho}{\partial x_3}. \tag{2}$$

According to the convention of Einstein, the repetition of the index i implies a summation over this *dead* index. Substitution of definition (2) into equation (1) yields the mass conservation law applied to a fixed infinitesimal volume element:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \ \text{ or } \ \frac{\partial\rho}{\partial t} + \frac{\partial\rho v_i}{\partial x_i} = 0. \tag{3}$$

We call this the conservation form of the mass equation. For convenience one can introduce a mass source term Q_m in this equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = Q_m. \tag{4}$$

In a non-relativistic approximation such a mass source term is of course zero, and only introduced to represent the influence on the flow of a complex phenomenon (such as combustion) within the framework of a model that ignore the details of this process. Therefore, there is some ambiguity in the definition of Q_m. We should actually specify whether the injected mass has momentum and whether it has a different thermodynamic state than the surrounding fluid.

In agreement with the non-relativistic approximation we apply the second law of Newton to a fluid particle:

$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla \cdot \boldsymbol{\mathcal{P}} + \boldsymbol{f} \tag{5}$$

where $\boldsymbol{f}$ is the density of the force field acting on the bulk of the fluid, while $-\nabla \cdot \boldsymbol{\mathcal{P}}$ is the net force acting on the surface of the infinitesimal volume element. This force is expressed in terms of a stress tensor $\boldsymbol{\mathcal{P}}$. Using the mass-conservation law (3) without mass source term ($Q_m = 0$) we obtain the momentum equation in conservation form:

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{\mathcal{P}} + \rho \boldsymbol{v}\boldsymbol{v}) = \boldsymbol{f}, \quad \text{or} \quad \frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = -\frac{\partial \mathcal{P}_{ij}}{\partial x_j} + f_i. \tag{6}$$

The isotropic part $p\delta_{ij}$ of this tensor corresponds to the effect of the hydrodynamic pressure $p = \mathcal{P}_{ii}/3$:

$$\mathcal{P}_{ij} = p\delta_{ij} - \sigma_{ij} \tag{7}$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. The deviation σ_{ij} from the hydrostatic behaviour corresponds in a simple fluid to the effect of viscosity. We define a simple fluid as a fluid for which σ_{ij} is symmetrical [3].

The energy equation applied to a material element is:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(e + \tfrac{1}{2}v^2) = -\nabla \cdot \boldsymbol{q} - \nabla \cdot (\boldsymbol{\mathcal{P}} \cdot \boldsymbol{v}) + \boldsymbol{f} \cdot \boldsymbol{v} + Q_w \tag{8}$$

where e is the internal energy per unit of mass, $v = \|\boldsymbol{v}\|$, $\boldsymbol{q}$ the heat flux and Q_w is the heat production per unit of volume. In conservation form this equation becomes:

$$\frac{\partial}{\partial t}\rho(e + \tfrac{1}{2}v^2) + \nabla \cdot [\rho \boldsymbol{v}(e + \tfrac{1}{2}v^2)] = -\nabla \cdot \boldsymbol{q} - \nabla \cdot (\boldsymbol{\mathcal{P}} \cdot \boldsymbol{v}) + \boldsymbol{f} \cdot \boldsymbol{v} + Q_w, \tag{9a}$$

or in index notation

$$\frac{\partial}{\partial t}\rho(e + \tfrac{1}{2}v^2) + \frac{\partial}{\partial x_i}\left[\rho v_i(e + \tfrac{1}{2}v^2)\right] = -\frac{\partial q_i}{\partial x_i} - \frac{\partial \mathcal{P}_{ij} v_j}{\partial x_i} + f_i v_i + Q_w. \tag{9b}$$

The mass, momentum and energy conservation laws in differential form are only valid when the derivatives of the flow variables are defined. When those laws are applied to a finite volume V one obtains integral formulations which are also valid in the presence of discontinuities such as shock waves. For an arbitrary volume V, enclosed by a surface S with outer normal $\boldsymbol{n}$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V + \int_S \rho(\boldsymbol{v}-\boldsymbol{b})\cdot\boldsymbol{n}\,\mathrm{d}S = 0, \tag{10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\boldsymbol{v}\,\mathrm{d}V + \int_S \rho\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{b})\cdot\boldsymbol{n}\,\mathrm{d}S = -\int_S \boldsymbol{\mathcal{P}}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_V \boldsymbol{f}\,\mathrm{d}V \tag{10b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(e+\tfrac{1}{2}v^2)\,\mathrm{d}V + \int_S \rho(e+\tfrac{1}{2}v^2)(\boldsymbol{v}-\boldsymbol{b})\cdot\boldsymbol{n}\,\mathrm{d}S = -\int_S \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_S (\boldsymbol{\mathcal{P}}\cdot\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}S + \int_V \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}V \tag{10c}$$

where $\boldsymbol{b}$ is the velocity of the control surface S. For a material control volume we have $\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{b}\cdot\boldsymbol{n}$. For a fixed control volume we have $\boldsymbol{b} = 0$.

2.2 Constitutive equations

The mass, momentum and energy equations (3), (6) and (9) involve much more unknowns than equations. The additional information needed to obtain a complete set of equations is provided by empirical information in the form of constitutive equations. An excellent approximation is obtained by assuming the fluid to be locally in thermodynamic equilibrium, i.e. within a material element [61]. This implies for a homogeneous fluid that two intrinsic state variables fully determine the state of the fluid. For acoustics it is convenient to choose the density of mass ρ and the specific entropy (i.e. per unit of mass) s as variables. All other intrinsic state variables are function of ρ and s. Hence the specific energy e is completely defined by a relation

$$e = e(\rho, s). \tag{11}$$

This is what we call a thermal equation of state. This equation is determined empirically. Variations of e may therefore be written as

$$\mathrm{d}e = \left(\frac{\partial e}{\partial \rho}\right)_s \mathrm{d}\rho + \left(\frac{\partial e}{\partial s}\right)_\rho \mathrm{d}s. \tag{12}$$

Comparison with the fundamental equation of thermodynamics,

$$\mathrm{d}e = T\mathrm{d}s - p\mathrm{d}\rho^{-1}, \tag{13}$$

provides thermodynamic equations for the temperature T and the pressure p:

$$T = \left(\frac{\partial e}{\partial s}\right)_\rho \tag{14}$$

and:

$$p = \rho^2\left(\frac{\partial e}{\partial \rho}\right)_s. \tag{15}$$

As p is also a function of ρ and s we have:

$$\mathrm{d}p = \left(\frac{\partial p}{\partial \rho}\right)_s \mathrm{d}\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \mathrm{d}s. \tag{16}$$

As sound is defined as isentropic ($\mathrm{d}s = 0$) pressure-density perturbations, the speed of sound $c = c(\rho, s)$ is defined by:

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}. \tag{17}$$

An extensive discussion of the speed of sound in air and water is provided by Pierce [102]. In many applications the fluid considered is air at ambient pressure and temperature. Under such conditions we can assume the ideal gas law to be valid:

$$p = \rho R T \tag{18}$$

where R is the specific gas constant, the ratio $R = k_B/m_w$ of the constant of Boltzmann k_B and of the mass of a molecule m_w. By definition, for such an ideal gas the energy density only depends on T, $e = e(T)$, and we have:

$$c = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma R T} \tag{19}$$

where $\gamma = c_p/c_v$ is the Poisson ratio of the specific (i.e. per unit of mass) heat capacities at respectively constant volume:

$$c_v = \left(\frac{\partial e}{\partial T}\right)_\rho \tag{20}$$

and constant pressure:

$$c_p = \left(\frac{\partial i}{\partial T}\right)_p \tag{21}$$

where i is the enthalpy per unit of mass defined by:

$$i = e + \frac{p}{\rho}. \tag{22}$$

For an ideal gas we have $c_p - c_v = R$. An ideal gas with constant specific heats is called a perfect gas.

As we consider local thermodynamic equilibrium, it is reasonable [61, 126] to assume that transport processes are determined by linear functions of the gradients of the flow state variables. This corresponds to a Newtonian fluid behaviour:

$$\sigma_{ij} = 2\eta\left(D_{ij} - \tfrac{1}{3} D_{kk}\delta_{ij}\right) + \mu_v D_{kk}\delta_{ij} \tag{23}$$

where the rate of strain tensor D_{ij} is defined by:

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{24}$$

Note that $D_{kk} = \boldsymbol{\nabla} \cdot \boldsymbol{v}$ takes into account the effect of dilatation. In thermodynamic equilibrium, according to the hypothesis of Stokes, one assumes that the bulk viscosity μ_v vanishes. The dynamic viscosity η is a function of the thermodynamic state of the fluid. For an ideal gas η is a function of the temperature only. While the assumption of vanishing bulk viscosity μ_v is initially an excellent approximation, one observes significant effects of the bulk viscosity in acoustical applications such as propagation over large distances [102]. This deviation from local thermodynamic equilibrium is due in air to the finite relaxation time of rotational degrees of freedom of molecules. The corresponding approximation for the heat flux $\boldsymbol{q}$ is the law of Fourier:

$$q_i = -K \frac{\partial T}{\partial x_i} \tag{25}$$

where K is the heat conductivity. For an ideal gas, K is a function of the temperature only. It is convenient to introduce the kinematic viscosity ν and the heat diffusivity a:

$$\nu = \frac{\eta}{\rho} \tag{26}$$

and

$$a = \frac{K}{\rho c_p}. \tag{27}$$

The kinematic viscosity and the heat diffusivity are diffusion coefficients for respectively momentum and heat transfer. For an ideal gas both transfer processes are determined by the same molecular velocities and similar free molecular path. This explains why the Prandtl number $Pr = \nu/a$ is of order unity. For air at ambient pressure and temperature $Pr = 0.72$.

2.3 Approximations and alternative forms of the basic equations

Starting from the energy equation (9) and using the thermodynamic law (13) one can derive an equation for the entropy:

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = -\nabla\cdot\boldsymbol{q} + \boldsymbol{\sigma}:\nabla\boldsymbol{v} + Q_w. \tag{28}$$

If heat transfer and viscous dissipation are negligible and there are no heat sources, the entropy equation reduces to:

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0. \tag{29}$$

Hence the entropy of a material element remains constant and the flow is isentropic. When the entropy is uniform we call the flow homentropic, so $\nabla s = 0$. An isentropic flow originating from a reservoir with uniform state is homentropic.

When there is no source of entropy, the sound generation is dominated by the fluctuations of the Reynolds stress $\rho v_i v_j$. Sound generation corresponds therefore often to conditions for which the term $|\partial \rho v_i v_j/\partial x_j|$ in the momentum equation (6) is large compared to $|\partial \sigma_{ij}/\partial x_j|$. Assuming that both gradients scale with the same length D while the velocity scales with U_0 (a " main flow velocity")we find: $Re = U_0 D/\nu \gg 1$, where Re is the Reynolds number. In such case one can also show that the dissipation is limited to thin boundary layers near the wall and that for time scales of the order of U_0/D the bulk of the flow can be considered as isentropic. Note that the demonstration of this statement in aero-acoustics has been subject of research for a long time [79, 80, 94, 137]. It is not a trivial statement. Actually, a turbulent flow is essentially dissipative. On the time scales relevant to sound production dissipation is negligible outside the viscous boundary layers at walls [80]. We further often assume that heat transfer is limited to thin boundary layers at the wall and that the bulk of the flow is essentially isothermal. We will see later, that, when the entropy of the flow is not uniform, the convection of those inhomogeneities is an important source of sound. We have now discussed the problem of dissipation and heat transfer in the source region. We will later consider the effect of friction and heat transfer on wave propagation.

In a frictionless flow the momentum equation (6) reduces to the equation of Euler:

$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p + \boldsymbol{f}. \tag{30}$$

Using the definition of enthalpy (22): $i = e + p/\rho$ combined with the fundamental equation (13): $T\mathrm{d}s = \mathrm{d}e + p\mathrm{d}(1/\rho)$ we find:

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla i + T\nabla s + \frac{\boldsymbol{f}}{\rho}. \tag{31}$$

The acceleration $\mathrm{D}\boldsymbol{v}/\mathrm{D}t$ can be split up into an effect of the time dependence of the flow $\partial \boldsymbol{v}/\partial t$, an acceleration in the direction of the streamlines $\nabla(\frac{1}{2}v^2)$ and a Corriolis acceleration due to the rotation $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ of the fluid as follows:

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\partial \boldsymbol{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) + \boldsymbol{\omega} \times \boldsymbol{v}. \tag{32}$$

Substitution of (32) and (31) in Euler's equation (30) yields:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla B = -\boldsymbol{\omega} \times \boldsymbol{v} + T\nabla s + \frac{\boldsymbol{f}}{\rho} \tag{33}$$

where the total enthalpy or Bernoulli constant B is defined by:

$$B = i + \tfrac{1}{2}v^2. \tag{34}$$

In general, the flow velocity field $\boldsymbol{v}$ can be expressed in terms of a scalar potential ϕ and a vector stream-function $\boldsymbol{\psi}$:

$$\boldsymbol{v} = \nabla\phi + \nabla \times \boldsymbol{\psi}. \tag{35}$$

There is an ambiguity in this definition, which may be removed by some additional condition. One can for example impose $\nabla \cdot \boldsymbol{\psi} = 0$. In most of the problems considered the ambiguity is removed by boundary conditions imposed on ϕ and $\boldsymbol{\psi}$. While the scalar potential ϕ is related to the dilatation rate

$$\nabla \cdot \boldsymbol{v} = \nabla^2 \phi \tag{36}$$

because $\nabla \cdot (\nabla \times \boldsymbol{\psi}) = 0$, the vector stream function $\boldsymbol{\psi}$ is related to the vorticity:

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \nabla \times (\nabla \times \boldsymbol{\psi}) \tag{37}$$

because $\nabla \times \nabla\phi = 0$. This will be used as an argument to introduce the unsteady component of the potential velocity field as a definition for the acoustical field within the framework of Vortex Sound Theory (see section 4.5).

For a homentropic ($\nabla s = 0$) potential flow ($\boldsymbol{v} = \nabla\phi$) without external forces ($\boldsymbol{f} = 0$), the momentum equation 30) can be integrated to obtain the equation of Bernoulli:

$$\frac{\partial \phi}{\partial t} + B = g(t) \tag{38}$$

where the function $g(t)$ can be absorbed into the definition of the potential ϕ without any loss of generality.

In the Vortex Sound Theory (4.5), where the reference flow (acoustic field) corresponds to the unsteady component of the potential flow $\nabla\phi$, the source is the difference between the actual flow and the potential flow. Therefore, the sources are directly related to the vorticity $\boldsymbol{\omega}$. In a homentropic flow the density is a function $\rho = \rho(p)$ of the pressure p only (barotropic fluid). In such a case we can eliminate the pressure from the equation of Euler by taking the curl of this equation (30). We obtain an equation for the vorticity [124]:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{v} - \boldsymbol{\omega}\nabla\cdot\boldsymbol{v} + \nabla\times\left(\frac{\boldsymbol{f}}{\rho}\right). \tag{39}$$

In the absence of external forces, the equation reduces to a purely kinematic equation. Solving this equation yields, with $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$, the velocity field. This approach is most effective for two-dimensional plane flows $\boldsymbol{v} = (v_1(x_1, x_2), v_2(x_1, x_2), 0)$. In that case the vorticity equation reduces to $\mathrm{D}\omega_3/\mathrm{D}t = 0$. The study of such flows provides much insight into the behaviour of vorticity near sharp edges.

The assumed absent viscosity yields mathematically a set of equations and boundary conditions that have no unique solution. By adding the empirically observed condition that no vorticity is produced anywhere, we have again a unique solution. This, however, is not exactly true near sharp edges. Depending on the Reynolds number and the (dimensionless) frequency and amplitude, a certain amount of vorticity is shed from a sharp edge. For high enough Reynolds number and low enough frequency and amplitude, the amount of shed vorticity is just enough to remove the singularity of the potential flow around the edge. This is the so-called Kutta condition [63, 105, 99, 15, 110].

When the flow is nearly incompressible (such as in acoustical waves), we can approximate the enthalpy by:

$$i = \int \frac{\mathrm{d}p}{\rho} \simeq \frac{p}{\rho_0}. \tag{40}$$

Under these circumstances the equation of Bernoulli (38) reduces to:

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}v^2 + \frac{p}{\rho_0} = 0. \tag{41}$$

When considering acoustical waves propagating in a uniform stagnant medium we may neglect the quadratically small term $\frac{1}{2}v^2$, which yields the linearized equation of Bernoulli:

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho_0} = 0. \tag{42}$$

3 Free space acoustics of a quiescent fluid

3.1 Order of magnitudes

In acoustics one considers small perturbations of a flow. This will allow us to linearize the conservation laws and constitutive equations described in the previous section (2). We will focus here on acoustic perturbations of a uniform stagnant (quiescent) fluid. For that particular case we will now discuss order of magnitudes of various effects. This will justify the approximations which we use further on.

We will focus on the pressure perturbations p' which propagate as waves and which can be detected by the human ear. For harmonic pressure fluctuations the audio range is:

$$20\ \text{Hz} \leq f \leq 20\ \text{kHz}. \tag{43}$$

The Sound Pressure Level (SPL) measured in decibel (dB) is defined by:

$$\text{SPL} = 20 \log_{10}\left(\frac{p'_{\text{rms}}}{p_{\text{ref}}}\right) \tag{44}$$

where $p_{\text{ref}} = 2 \times 10^{-5}$ Pa for sound propagating in gasses and $p_{\text{ref}} = 10^{-6}$ Pa for propagation in other media. The sound intensity $\langle I \rangle = \langle \boldsymbol{I} \cdot \boldsymbol{n} \rangle$ is defined as the time averaged energy flux associated to the acoustic wave, propagating in direction $\boldsymbol{n}$. The intensity level (IL) measured in decibel (dB) is given by:

$$\text{IL} = 10 \log_{10}\left(\frac{\langle I \rangle}{I_{\text{ref}}}\right) \tag{45}$$

where in air $I_{\text{ref}} = 10^{-12}$ Wm^{-2}. The reference intensity level I_{ref} is related to the reference pressure p_{ref} by the relationship valid for propagating plane waves:

$$\langle I \rangle = \frac{p'^2}{\rho_0 c_0}, \tag{46}$$

because in air at ambient conditions $\rho_0 c_0 \simeq 400$ kg m^{-2}s^{-1}. The time averaged power $\langle P \rangle$ generated by a sound source is the flux integral of the intensity $\langle \boldsymbol{I} \rangle$ over a surface enclosing the source. The Sound Power Level (PWL) measured in decibel (dB) is defined by:

$$\text{PWL} = 10 \log_{10}\left(\frac{\langle P \rangle}{P_{\text{ref}}}\right) \tag{47}$$

where $P_{\text{ref}} = 10^{-12}$ W corresponds to the power flowing through a surface of 1 m^2 surface area with an intensity $\langle I \rangle = I_{\text{ref}}$.

The threshold of hearing (for good ears) at 1 kHz is typically around SPL $= 0$ dB. This corresponds physically to the thermal fluctuations in the flux of molecules colliding with our eardrum. In order to detect 1 kHz we can at most integrate the signal over about 0.5 ms. At ambient conditions this corresponds to the collision of $N \simeq 10^{20}$ molecules with our eardrum. The thermal fluctuations in the measured pressure is therefore of the order of $p_0/\sqrt{N} = 10^{-5}$ Pa, with p_0 the atmospheric pressure. The maximum sensitivity of the ear is around 3 kHz (pitch of a police man whistle). Which is due to the quarter-wave-length resonance of our outer ear, a channel of about 2.5 cm depth. The threshold of pain is around SPL=140 dB. Even at such high levels we have pressure fluctuations only of the order $p'/p_0 = \mathcal{O}(10^{-3})$. The corresponding density fluctuations are:

$$\frac{\rho'}{\rho_0} = \frac{p'}{\rho_0 c_0^2} \tag{48}$$

also of the order of 10^{-3}, because in air $\rho_0 c_0^2/p_0 = \gamma = c_p/c_v \simeq 1.4$. This justifies the linearisation of the equations. Note that in a liquid the condition for linearisation $\rho'/\rho_0 \ll 1$ does not imply a small value of the pressure fluctuations because $p'/p_0 = (\rho_0 c_0^2/p_0)(\rho'/\rho_0)$ while $\rho_0 c_0^2 \gg p_0$. In water $\rho_0 c_0^2 = 2 \times 10^9$ Pa. We should note, that when considering wave propagation over large distances nonlinear wave steepening will play a significant role. In a pipe this can easily result into the formation of shock waves. This explains the occurrence of brassy sound in trombones at fortissimo levels [38, 40]. Also in sound generated by aircraft nonlinear wave distortion significantly contributes to the spectral distribution [16].

For a propagating acoustic plane wave the pressure fluctuations p' are associated to the velocity u' of fluid particles in the direction of propagation. We will see later that:

$$u' = \frac{p'}{\rho_0 c_0}. \tag{49}$$

The amplitude δ of the fluid particle displacement is for a harmonic wave with circular frequency ω given by: $\delta = |u'|/\omega$. At $f = 1$ kHz the threshold of hearing (0 dB) corresponds with $\delta = 10^{-11}$ m. At the threshold of pain we find $\delta = 10^{-4}$ m. Such small displacements also justify the use of a linearized theory. When the acoustical displacement δ becomes of the same order of magnitude as the radius of curvature of a wall, one will observe acoustical flow separation and the formation of vortices. In a pipe when δ approaches the pipe cross-sectional radius one will observe acoustical streaming. At the pipe outlet this will result into periodic vortex shedding [52, 53, 22, 101, 100, 18]. In woodwind-musical instruments and bas-reflex ports of loudspeaker boxes this is a common phenomenon [38, 18, 123].

When deriving a wave equation in the next section, we will not only linearize the basic equations, but we will also neglect friction and heat-transfer. This corresponds to the assumption that in an acoustical wave, with wave-length $\lambda = c/f$, the unsteady Reynolds number:

$$Re_{\text{unst}} = \frac{\lambda^2 f}{\nu} = \mathcal{O}\Big(\frac{|\rho \frac{\partial u}{\partial t}|}{|\eta \frac{\partial^2 u}{\partial x^2}|}\Big), \tag{50}$$

is very large. For air $\nu = 1.5 \times 10^{-5}\ \text{m}^2\text{s}^{-1}$ so that for $f = 1$ kHz we find $Re_{\text{unst}} = \mathcal{O}(10^7)$. We therefore expect that viscosity only plays a role on very large distances. As the Prandtl number is of order unity $Pr = \mathcal{O}(1)$ in a gas, we expect heat transfer to be also negligible. At high frequencies we however observe a much stronger attenuation due to non-equilibrium effects (bulk-viscosity). This results in a strong absorption of these high frequencies when we listen to aircraft at large distances. Furthermore, in the presence of walls visco-thermal dissipation will also be much larger. The amplitude of a plane wave travelling along a tube of cross-sectional radius R will attenuate exponentially $\exp(-\alpha x)$ with the distance x. The attenuation coefficient is given for typical audio-conditions by [102, 125]:

$$\alpha = \frac{\sqrt{\pi f \nu}}{R c_0}\Big(1 + \frac{\gamma - 1}{\sqrt{Pr}}\Big). \tag{51}$$

In most woodwind musical instruments at low pitches the visco-thermal dissipation losses are larger than the sound radiation power [34].

3.2 Wave equation and sources of sound

We consider the propagation of pressure perturbations p' in an otherwise quiescent fluid. The perturbation of the uniform constant reference state p_0, ρ_0, s_0, $\boldsymbol{v}_0$ are defined by:

$$p' = p - p_0, \quad \rho' = \rho - \rho_0, \quad s' = s - s_0, \quad \boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{v}_0, \tag{52}$$

where for a quiescent fluid $\boldsymbol{v}_0 = 0$. We assume that $\boldsymbol{f}$, Q_w and the perturbations p'/p_0, ρ'/ρ_0, ... are small so that we can linearize the basic equations. We neglect furthermore heat transfer and viscous effects. The equations of motion (3, 6 and 28)reduce to:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla\cdot \boldsymbol{v}' = 0, \qquad \rho_0 \frac{\partial \boldsymbol{v}'}{\partial t} + \nabla p' = \boldsymbol{f}, \qquad \rho_0 T_0 \frac{\partial s'}{\partial t} = Q_w, \tag{53}$$

and the constitutive equation (17) becomes:

$$p' = c_0^2 \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s'. \tag{54}$$

Subtracting the divergence of the linearized momentum equation from the time derivative of the linearized mass-conservation law yields:

$$\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = -\nabla\cdot \boldsymbol{f}. \tag{55}$$

Combining the entropy equation with the constitutive equation yields:

$$\frac{\partial^2 p'}{\partial t^2} = c_0^2 \frac{\partial^2 \rho'}{\partial t^2} + \frac{(\partial p/\partial s)_\rho}{\rho_0 T_0} \frac{\partial Q_w}{\partial t}. \tag{56}$$

Elimination of the density fluctuations from equations (55) and (56) yields a non-homogeneous wave-equation:

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = q, \tag{57}$$

$$q = \frac{(\partial p/\partial s)_\rho}{\rho_0 c_0^2 T_0} \frac{\partial Q_w}{\partial t} - \nabla\cdot \boldsymbol{f}.$$

The first source term corresponds to the dilatation of the fluid as a result of heat production in processes such as unsteady combustion or condensation. This type of sound generation mechanism have been discussed in detail by Morfey [78] and Dowling [16]. The second term describes the sound production by a non-uniform unsteady external force field.When considering a moving body, the reaction of the body to the force exerted by the fluid can be represented by such a force field. An example of this is a model of the sound radiated by a rotor calculated by concentrating the lift force of each wing into a point force. This model will be discussed later and corresponds to the first theory of sound generation of propellers as formulated by Gutin [36] and commonly used in many applications [9, 122].

We introduced $q(\boldsymbol{x}, t)$ as shorthand notation for the source term in the wave equation. In the absence of a source term, $q = 0$, the sound field is due to initial perturbations or boundary conditions. In the next section we present a general solution of the wave equation.

3.3 Green's function and integral formulation

Using Green's theorem [81] we can obtain an integral equation which includes the effects of the sources, the boundary conditions and the initial conditions on the acoustic field. The Green's function $G(\boldsymbol{x}, t|\boldsymbol{y}, \tau)$ is defined as the response of the flow to a impulsive point source represented by delta functions of space and time:

$$\frac{1}{c_0^2} \frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{y})\delta(t - \tau) \tag{58}$$

where $\delta(\boldsymbol{x}-\boldsymbol{y})=\delta(x_1-y_1)\delta(x_2-y_2)\delta(x_3-y_3)$. The delta function $\delta(t)$ is not a common function with a pointwise meaning, but a generalised function [16] formally defined by its filter property:

$$\int_{-\infty}^{\infty} F(t)\delta(t)\,\mathrm{d}t = F(0) \tag{59}$$

for any well-behaving function $F(t)$.

The definition of the Green's function G is completed by specifying boundary conditions at a surface S with outer normal $\boldsymbol{n}$ which encloses both the source placed at position $\boldsymbol{y}$ and the observer placed at position $\boldsymbol{x}$. As we consider here an acoustical phenomenon we follow Crighton's[1] suggestion to call the observer a *listener*. A quite general linear boundary condition is a linear relationship between the value of the Green's function G at the surface S and the (history of the) gradient $\boldsymbol{n}\cdot\nabla G$ at the same point. If this relationship is a property of the surface and independent of G, we call the surface locally reacting. Such a boundary condition is usually expressed in Fourier domain in terms of an impedance $Z(\omega)$ of the surface S, i.e. the ratio between pressure and normal velocity component, as follows:

$$\frac{\mathrm{i}}{\omega\rho_0}Z(\omega) = \frac{\hat{G}}{\boldsymbol{n}\cdot\nabla_{\boldsymbol{x}}\hat{G}} \tag{60}$$

where $\hat{G}$ is the Fourier transformed Green's function defined by

$$\hat{G}(\boldsymbol{x},\omega|\boldsymbol{y},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\boldsymbol{x},t|\boldsymbol{y},\tau)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{61}$$

and its inverse:

$$G(\boldsymbol{x},t|\boldsymbol{y},\tau) = \int_{-\infty}^{\infty} \hat{G}(\boldsymbol{x},\omega|\boldsymbol{y},\tau)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega. \tag{62}$$

(Always check the sign convention in the exponential! Here, we used $\exp(+\mathrm{i}\omega t)$. This is not essential as long as the same convention is used throughout!) A problem when using Fourier analysis is that the causality of the solution is not self-evident. We need to impose restrictions on the functional dependence of Z and $1/Z$ on the frequency ω [115, 119].

Causality implies that there is no response before the pulse $\delta(\boldsymbol{x}-\boldsymbol{y})\delta(t-\tau)$ has been released, so

$$G(\boldsymbol{x},t|\boldsymbol{y},\tau) = 0 \qquad \text{for } t<\tau. \tag{63}$$

Consider a Green's function G, not necessarily satisfying the actual boundary condition prevailing on S and a source q, not necessarily vanishing before some time t_0. For the wave equation (57) we find then the formal solution:

$$\begin{aligned} p'(\boldsymbol{x},t) = &\int_{t_0}^{t}\int_V q(\boldsymbol{y},\tau)G(\boldsymbol{x},t|\boldsymbol{y},\tau)\,\mathrm{d}V_y\,\mathrm{d}\tau \\ &+\int_{t_0}^{t}\int_S \Big(G(\boldsymbol{x},t|\boldsymbol{y},\tau)\nabla_y p' - p'(\boldsymbol{y},t)\nabla_y G\Big)\cdot\boldsymbol{n}\,\mathrm{d}S_y\,\mathrm{d}\tau \\ &+\frac{1}{c_0^2}\int_V\left[G(\boldsymbol{x},t|\boldsymbol{y},\tau)\frac{\partial p'}{\partial\tau} - p'(\boldsymbol{y},\tau)\frac{\partial G}{\partial\tau}\right]_{\tau=t_0}\mathrm{d}V_y \end{aligned} \tag{64}$$

[1] Crighton D.G., private communication (1992).

where $dV_y = dy_1 dy_2 dy_3$. The first integral is the convolution of the source q with the pulse response G, the Green's function. The second integral represents the effect of differences between the actual physical boundary conditions on the surface S and the conditions applied to the Green's function. When the Green's function satisfies the same locally reacting linear boundary conditions as the actual field, this surface integral vanishes. In that case we say that the Green's function is "tailored". The last integral represents the contribution of the initial conditions at t_0 to the acoustic field. If $q = 0$ and $p' = 0$ before some time, we can choose $t_0 = -\infty$ and leave this term out.

Note that in the derivation of the integral equation (64) we have make use of the reciprocity relation for the Green's function [81]:

$$G(\boldsymbol{x}, t|\boldsymbol{y}, \tau) = G(\boldsymbol{y}, -\tau|\boldsymbol{x}, -t). \tag{65}$$

Due to the symmetry of the wave operator considered, the acoustical response measured in $\boldsymbol{x}$ at time t of a source placed in $\boldsymbol{y}$ fired at time τ is equal to the response measured in $\boldsymbol{y}$ at time $-\tau$ of a source placed in $\boldsymbol{x}$ fired at time $-t$. The change of sign of the time $t \to -\tau$ and $\tau \to -t$ is necessary to respect causality. The reciprocity relation will be used later to determine the low frequency approximation of a tailored Green's function. This method is extensively used by Howe [47, 48]. It is a particularly powerful method for flow near a discontinuity at a wall. In many cases, however, it is more convenient to use a very simple Green's function such as the free-space Green's function G_0. We will introduce this Green's function after we have obtained some elementary solutions of the homogeneous wave equation in free space.

3.4 Inverse problem and uniqueness of source

It can be shown that for given boundary conditions and sources $q(\boldsymbol{x}, t)$ the wave equation has a unique solution [81]. However, different sources can produce the same acoustical field. A good audio system is able to produce a music performance that is just as realistic as the original. Mathematically the non-uniqueness of the source is demonstrated by the following enlightening example of Ffowcs Williams [25]. Let us assume that $p'(\boldsymbol{x}, t)$ is a solution of the non-homogeneous wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = q(\boldsymbol{x}, t) \tag{66}$$

in which $q(\boldsymbol{x}, t) \neq 0$ in a limited volume V. Outside V the source vanishes, so $q(\boldsymbol{x}, t) = 0$. As a result, $p' + q = p'$ for any $\boldsymbol{x} \notin V$. However, $p' + q$ satisfies the equation:

$$\frac{1}{c_0^2}\frac{\partial^2 (p'+q)}{\partial t^2} - \nabla^2 (p'+q) = q(\boldsymbol{x}, t) + \frac{1}{c_0^2}\frac{\partial^2 q}{\partial t^2} - \nabla^2 q \tag{67}$$

which has in general a different source term than equation (66).

In order to determine the source from any measured acoustical field outside the source region, we need a physical model of the source. This is typical of any inverse problem in which the solution is not unique. When using microphone arrays to determine the sound sources responsible for aircraft noise one usually assumes that the sound field is built up of so-called monopole sound sources [129]. We will see later that the sound sources are more accurately described in terms of dipoles or quadrupoles (section 4.1). Under such circumstances it is hazardous to extrapolate such a monopole model to angles outside the measuring range of the microphone array or to the field from flow Mach numbers other than used in the experiments.

3.5 Elementary solutions of the wave equation

We consider two elementary solutions of the homogeneous wave equation ($q = 0$):

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0 \tag{68}$$

which will be used as building blocks to obtain more complex solutions:

- the plane wave
- the spherical symmetric wave.

We assume in both cases that these waves have been generated by some boundary condition or initial condition. We consider their propagation through an in all directions infinitely large quiescent fluid, which we call “free space”.

We first consider plane waves. These are uniform in any plane normal to the direction of propagation. Let us assume that the waves propagate in the x_1-direction, in which case $p' = p'(x_1, t)$ and the wave equation reduces to:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_1^2} = 0. \tag{69}$$

This 1D wave equation has the solution of d’Alembert:

$$p' = \mathcal{F}\Big(t - \frac{x_1}{c_0}\Big) + \mathcal{G}\Big(t + \frac{x_1}{c_0}\Big), \tag{70}$$

where $\mathcal{F}$ represents a wave travelling in the positive x_1 direction and G travels in the opposite direction. This result is easily verified by applying the chain rule for differentiation. The functions $\mathcal{F}$ and $\mathcal{G}$ are determined by the initial and boundary conditions.

Consider for example the acoustic field generated by an infinite plane wall oscillating around $x_1 = 0$ with a velocity $u_0(t)$ in the x_1 direction. In linear approximation $v_1'(0, t) = u_0(t)$, i.e. the acoustical velocity at $x_1 = 0$ is assumed to be equal to the wall velocity. It is furthermore implicitly assumed in the definition of “free-space” that no waves are generated at infinity. Therefore we have for $x_1 > 0$ that $\mathcal{G} = 0$. Using the linearized equation of motion (53) in the absence of external force field $\boldsymbol{f} = 0$:

$$\rho_0\frac{\partial v_1'}{\partial t} = -\frac{\partial p_1'}{\partial x_1} \tag{71}$$

we find:

$$p' = \rho_0 c_0 v_1'. \tag{72}$$

We call $\rho_0 c_0$ the specific acoustical impedance of the fluid. Using the boundary condition $v_1'(0, t) = u_0(t)$ and $p'(x_1, t) = \mathcal{F}(t - x_1/c_0)$ we find as solution:

$$p' = \rho_0 c_0 u_0(t - x_1/c_0) \tag{73}$$

for $x_1 > 0$. This equation states that perturbations, observed at time t at position x_1, are generated at the wall $x_1 = 0$ at time $t - x_1/c_0$. The time $t_e = t - x_1/c_0$ is called the emission time or retarded time. In a similar way we find:

$$p' = -\rho_0 c_0 u_0(t + x_1/c_0) \tag{74}$$

for $x_1 < 0$ if the wall is of zero thickness and perturbs the fluid at either side.

By analogy of (70), we easily find for a plane-wave solution propagating in a direction given by the unit vector $\boldsymbol{n}$ the most general form

$$p' = \mathcal{F}\Big(t - \frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c_0}\Big). \tag{75}$$

For the particular case of harmonic waves the plane wave solution is written in complex notation as:

$$p' = \hat{p}\,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \tag{76}$$

where $\boldsymbol{k} = k\boldsymbol{n}$ is the wave vector, $k = \omega/c_0$ is the wave number and $\hat{p}$ is the amplitude. The complex notation is a shorthand notation for:

$$p' = \mathrm{Re}(\hat{p}\,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}) = \mathrm{Re}(\hat{p})\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{x}) - \mathrm{Im}(\hat{p})\sin(\omega t - \boldsymbol{k}\cdot\boldsymbol{x}). \tag{77}$$

By means of Fourier analysis in time, an arbitrary time dependence can be represented by a sum or integral of harmonics functions. In a similar way general spatial distributions can be developed in terms of plane waves.

Another important elementary solution of the homogeneous wave equation (68) is the spherically symmetric wave. In that case the pressure is only a function $p'(r,t)$ of time and the distance r to the origin.

By identifying $\nabla^2 F(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial F}{\partial r}\big) = \frac{1}{r}\frac{\partial^2 rF}{\partial r^2}$, the wave equation (68) reduces for $r > 0$ to:

$$\frac{1}{c_0^2}\frac{\partial^2 p'r}{\partial t^2} - \frac{\partial^2 p'r}{\partial r^2} = 0. \tag{78}$$

Note that at $r = 0$ the equation is singular. As we will see this will correspond with a possible point source. Equation (78) implies that the product $p'r$ of the pressure p' and the radius r, satisfies the 1D wave equation, and may be expressed as a solution of d'Alembert:

$$p' = \frac{1}{r}\Big[\mathcal{F}(t - r/c_0) + \mathcal{G}(t + r/c_0)\Big] \tag{79}$$

in which $\mathcal{F}$ represents outgoing waves and $\mathcal{G}$ represents incoming waves. In many applications we will assume that there are no incoming waves $G = 0$. We call this *free-field* conditions. We now focus on the behaviour of outgoing harmonic waves:

$$p' = \frac{A}{r}\,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}kr} \tag{80}$$

where A is the amplitude and $k = \omega/c_0$ the wave number. The radial fluid particle velocity v_r' associated with the wave can be calculated by using the radial component of the momentum equation (53):

$$\rho_0\frac{\partial v_r'}{\partial t} = -\frac{\partial p'}{\partial r}. \tag{81}$$

We find:

$$v_r' = \frac{p'}{\rho_0 c_0}\Big(1 - \frac{i}{kr}\Big). \tag{82}$$

At distances r large compared to the wave length $\lambda = 2\pi/k$ ($kr = 2\pi r/\lambda \gg 1$) we find the same behaviour as for a plane wave (72). The spatial variation due to the harmonic wave motion dominates over the effect of the radial expansion. We call this the *far-field* behaviour. In contrast to this, we have for $kr \ll 1$ the *near field* behaviour in which the velocity v_r' is inversely proportional to the square of the distance r. This is indeed the expected incompressible flow behaviour. Over small distances the speed of sound is effectively infinite because any perturbation arrives without delay in time. As a result, the mass flux is conserved and $v_r' r^2$ is constant. All this can be understood by the observation that $|(\partial^2 p'/\partial t^2)/[c_0^2(\partial^2 p'/\partial^2 r^2)] \sim (kr)^2$ so that the wave equation reduces to the equation of Laplace $\nabla^2 p' = 0$ for $kr \to 0$.

Outgoing spherical symmetric waves correspond to what is commonly called a monopole sound field. Such a field can be generated by a harmonically pulsating rigid sphere with radius a:

$$a = a_0 + \hat{a}\, \mathrm{e}^{\mathrm{i}\omega t} \,. \tag{83}$$

In linear approximation (in $\hat{a}/a_0$) we have:

$$\hat{v}_r(a_0) = \mathrm{i}\omega\hat{a}. \tag{84}$$

Combining this boundary condition with equations (80) and (82) we obtain the amplitude A of the wave:

$$p' = -\frac{\rho_0\omega^2 a_0\hat{a}}{1+\mathrm{i}ka_0}\,\frac{a_0}{r}\,\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}k(r-a_0)} \,. \tag{85}$$

In the low frequency limit $ka_0 \ll 1$ we see that the amplitude of the radiated sound field decreases with the frequency. If the volume flux $\Phi_V = 4\pi a_0^2 v_r'(a_0) = 4\pi\,\mathrm{i} a_0^2\omega\hat{a}$, generated at the surface of the sphere, is kept fixed the sound pressure p' decreases linearly with decreasing frequency:

$$p' = \frac{\mathrm{i}\omega\rho_0\Phi_V}{4\pi r}\,\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}k(r-a_0)} \,. \tag{86}$$

A monopole field can for example be generated by unsteady combustion, which corresponds to the entropy source term in the wave equation. This will occur in particular for a spherically symmetric combustion. In general the monopole field will be dominant when the source region is small compared to the acoustic wave length $ka_0 \ll 1$. We call a region which is small compared to the wave length a *compact* region. We have seen that a compact pulsating sphere is a rather inefficient source of sound under free-field conditions. More formally, a monopole source corresponds to a localized volume source or point source placed at position $\boldsymbol{y}$:

$$q(\boldsymbol{x},t) = \frac{\partial\Phi_V}{\partial t}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{87}$$

We will discuss this approach more in detail later. Note the time derivative in the source term of equation (87): it reflects the fact that a steady flow does not produce any sound.

Using the monopole solution (80) we can build more complex solutions. If p_0' is a solution of the wave equation (68), any spatial derivatives $\partial p_0'/\partial x_i$ are also solutions because the wave equation has constant coefficients and the derivatives may be interchanged. A first order spatial derivative of the monopole field is called a dipole field. Second order spatial derivatives correspond to quadrupole fields.

An example of a dipole field is the acoustic field generated by a rigid sphere translating harmonically in a certain direction x_1 with a velocity $v_s = \hat{v}_s\, e^{i\omega t}$. The radial velocity $v_r'(a_0, \theta)$ on the surface of the sphere is given by:

$$\hat{v}_r(a_0, \theta) = \hat{v}_s \cos\theta \tag{88}$$

where θ is the angle between the position vector on the sphere and the translation direction x_1. Since we have the identity

$$\frac{\partial r}{\partial x_1} = \frac{\partial}{\partial x_1}\sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{x_1}{r} = \cos\theta, \tag{89}$$

we can write for the dipole field:

$$\hat{p} = A\frac{\partial}{\partial x_1}\Big(\frac{e^{-ikr}}{r}\Big) = A\cos\theta\frac{\partial}{\partial r}\Big(\frac{e^{-ikr}}{r}\Big). \tag{90}$$

Substitution of (90) into the momentum equation (81) yields:

$$i\omega\rho_0\hat{v}_r = -A\cos\theta\frac{\partial^2}{\partial r^2}\Big(\frac{e^{-ikr}}{r}\Big). \tag{91}$$

We apply this equation at $r = a_0$. Comparison with equation (88) yields:

$$\hat{p} = \frac{i\omega\rho_0\hat{v}_s a_0 \cos\theta}{2 + 2ika_0 - (ka_0)^2}(1 + ikr)\Big(\frac{a_0}{r}\Big)^2 e^{-ik(r-a_0)}\,. \tag{92}$$

Another example is the calculation of the field p' generated by an unsteady non-uniform force field $\boldsymbol{f} = (f_1, f_2, f_3)$. Following equation (57) we have

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = -\nabla\cdot\boldsymbol{f}. \tag{93}$$

Let us assume that we have obtained a solution F_1 of the wave equation in free space, thus satisfying

$$\frac{1}{c_0^2}\frac{\partial^2 F_1}{\partial t^2} - \nabla^2 F_1 = -f_1. \tag{94}$$

Then we may find the solution p' of equation (93) in free space by taking the space derivative of F_1

$$p' = \frac{\partial F_1}{\partial x_1}. \tag{95}$$

This indicates that the dipole field is related to forces exerted on the flow.

Another way to deduce the relationship between forces and dipole fields is to consider the dipole as the field obtained by placing two opposite monopole source of amplitude Φ_V at a distance Δy_1 from each other. Taking the limit of $\Delta y_1 \to 0$ while we keep $\Phi_V \Delta y_1$ constant yields a dipole field. As in free space changes in source position $\boldsymbol{y}$ are equivalent to changes in listener position $\boldsymbol{x}$, it is obvious that this limit relates to the spatial derivative of the monopole field. If we consider now the two oscillating volume sources forming the dipole, there will be a mass flow Φ_V from one source to the other. Such a unsteady mass flow is associated with an unsteady momentum flux. This unsteady momentum flux must, following Newton, be produced by an external force acting on the flow [105, 27]. Hence we see that a dipole is not possible without the action of a force. This idea is illustrated in figure 1 in which we consider waves generated by a boat on the water surface.

When a person jumps up and down in the boat, he produces an unsteady volume injection and this generates a monopole wave field around the boat. When two persons on the boat play with a ball, they will exert a force on the boat each time they throw or catch the ball. Exchanging the ball results into an oscillating force on the boat. This will make the boat translate and this generates a dipole wave field. We could say that two individuals fighting with each other is a reasonable model for a quadrupole. This indicates that quadrupoles are in general much less efficient in producing waves then monopoles or dipoles. This will indeed appear to be the case.

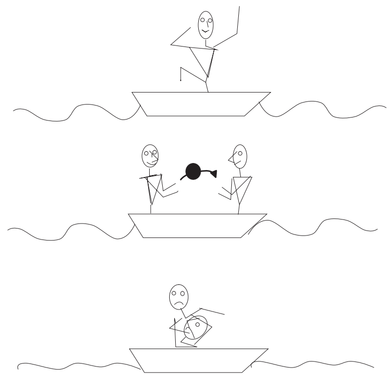

Figure 1: Monopole, dipole and quadrupole generating waves on the surface of the water around a boat.

It is often stated that Lighthill [69] has demonstrated that the sound produced by a free turbulent isentropic flow has the character of a quadrupole. A better way of putting it is that since in such flows there is no net volume injection due to entropy production nor any external force field, the sound field can at most be a quadrupole field [41]. Therefore, Lighthill's statement is actually that we should ignore any monopole or dipole emerging from a poor description of the flow. We will consider this later more in detail.

3.6 Acoustic energy and impedance

The definition of acoustical energy is not obvious when we define the acoustic field on the basis of linearized equations. The energy is essentially quadratic in the perturbations. We may anticipate therefore that there is some arbitrariness in the definition of acoustical energy. This problem has been the subject of many discussions in the literature [63, 102, 35, 76, 88, 89, 90, 55]. In the particular case of the acoustics of a quiescent fluid the approach proposed by Kirchhoff [63], starting from the linearized equations (53), appears to be equivalent to the result obtained by expanding the energy equation (9) up to the second order [63]. After elimination of the density by using the constitutive equation we can write the linearized mass conservation in the form:

$$\frac{1}{c_0^2}\frac{\partial p'}{\partial t} + \rho_0 \nabla\cdot\boldsymbol{v} = \frac{1}{c_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho \frac{\partial s'}{\partial t} \tag{96}$$

and the momentum equation in the form:

$$\rho_0 \frac{\partial \boldsymbol{v}'}{\partial t} + \nabla p' = \boldsymbol{f}. \tag{97}$$

We multiply the first equation (96) by p'/ρ_0 and add the result to the scalar product of the second equation (97) with $\boldsymbol{v}'$ to obtain the acoustic energy equation

$$\frac{\partial E}{\partial t} + \nabla\cdot\boldsymbol{I} = -\mathcal{D}, \tag{98}$$

where we have defined the acoustic energy E by:

$$E = \frac{1}{2}\rho_0 v'^2 + \frac{1}{2}\frac{p'^2}{\rho_0 c_0^2}. \tag{99}$$

The intensity $\boldsymbol{I}$, defined as

$$\boldsymbol{I} = p'\boldsymbol{v}', \tag{100}$$

is identified as the flux of acoustic energy. The dissipation $\mathcal{D}$ is the power per unit volume delivered by the acoustical field to the sources

$$\mathcal{D} = -\frac{1}{\rho_0 c_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho p' \frac{\partial s'}{\partial t} - \boldsymbol{f}\cdot\boldsymbol{v}'. \tag{101}$$

From the mass conservation law (96) we see that the source term $(\partial p/\partial s)_\rho/(\rho_0 c_0^2)(\partial s'/\partial t)$ in the dissipation, corresponds to the dilatation rate induced by the source. This allows us to relate the first term in the dissipation to the work of the acoustical field due to the change in volume ($\mathrm{d}W = p' \mathrm{d}V$).

For harmonically oscillating fields $p' = \hat{p}\,\mathrm{e}^{\mathrm{i}\omega t}$, $\boldsymbol{v}' = \hat{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\omega t}$ the time averaged $\langle E\rangle$ of the acoustic energy is (of course) independent of time

$$\langle E\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} E\,\mathrm{d}t, \tag{102}$$

hence the energy equation (98) reduces to:

$$\nabla\cdot\langle\boldsymbol{I}\rangle = -\langle\mathcal{D}\rangle. \tag{103}$$

By integration of this equation over a volume enclosing the sources we find the source power

$$\langle P\rangle = -\int_V \langle\mathcal{D}\rangle\,\mathrm{d}V = \int_S \langle\boldsymbol{I}\rangle\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{104}$$

where $\boldsymbol{n}$ is the outer normal to the control surface S. If we assume an impedance boundary condition on the surface S:

$$Z(\omega) = \frac{\hat{p}}{\hat{\boldsymbol{v}}\cdot\boldsymbol{n}} \tag{105}$$

we have:

$$\langle\boldsymbol{I}\rangle\cdot\boldsymbol{n} = \tfrac{1}{2}\,\mathrm{Re}(Z)|\hat{\boldsymbol{v}}\cdot\boldsymbol{n}|^2. \tag{106}$$

We see that the real part $\mathrm{Re}(Z)$ of the impedance Z is associated to the transport of acoustic energy through the surface S. The imaginary part is associated to pressure differences induced by the inertia of the flow.

We can now easily verify by using equation (104) that the spherically symmetric wave solution (80) satisfies the acoustic energy conservation law. The r^{-1} dependence of the pressure (79) in a simple outgoing wave results into a conserved value of $4\pi r^2\langle\boldsymbol{I}\rangle\cdot\boldsymbol{n}$.

To illustrate this we consider the impedance of a pulsating sphere of radius a_0. From equation (85) we find for the impedance Z of the surface of the sphere:

$$Z = \frac{\hat{p}}{\hat{v}_r} = \frac{\rho_0 c_0}{1 + \frac{1}{(ka_0)^2}}\left(1 + \frac{i}{ka_0}\right). \tag{107}$$

The real part is given by

$$\mathrm{Re}(Z) = \rho_0 c_0 \frac{(ka_0)^2}{1 + (ka_0)^2}. \tag{108}$$

We see that for a large sphere $ka_0 \gg 1$ the impedance is equal to $\rho_0 c_0$, the impedance experienced by a plane wave (72) of any plane control surface. For a compact sphere $ka_0 \ll 1$ we see that $\mathrm{Re}(Z) \simeq \rho_0 c_0 (ka_0)^2$ which implies very little energy transfer and so a very inefficient sound source. The imaginary part $\mathrm{Im}(Z)$ of the impedance of the sphere, given by

$$\mathrm{Im}(Z) = \rho_0 c_0 \frac{ka_0}{1 + (ka_0)^2}, \tag{109}$$

vanishes for $ka_0 \to \infty$. For a compact sphere, $ka_0 \ll 1$, it corresponds to the pressure calculated by means of the linearized equation of Bernoulli (42) if we assume an incompressible flow $\hat{v} = \mathrm{i}\omega\hat{a}(a_0/r)$ around the sphere. (Note that $\phi_\infty - \phi(a_0) = \int_{a_0}^{\infty} v_r \,\mathrm{d}r = \mathrm{i}\omega\hat{a}a_0$.)

Furthermore, we note that in order to deliver acoustical energy a volume source needs to be surrounded by a field of high pressure. This occurs when it is surrounded by a surface of which the real part of the impedance is large. A force field needs a large velocity fluctuation in order to produce acoustical energy efficiently. This corresponds to a large real part of the acoustical admittance $Y = 1/Z$.

3.7 Free space Green's function

The free space Green's function G_0 is the acoustical field generated at the observer's position $\boldsymbol{x}$ at time t by a pulse $\delta(\boldsymbol{x} - \boldsymbol{y})\delta(t - \tau)$ released in $\boldsymbol{y}$ at time τ. In order to calculate the free space Green's function G_0 we will make use of the Fourier transform (61, 62). We seek a spherically symmetric wave solution (80) of the form

$$\hat{G}_0 = \frac{A}{r}\,\mathrm{e}^{-\mathrm{i}kr} \qquad \text{where} \quad r = \|\boldsymbol{x} - \boldsymbol{y}\|. \tag{110}$$

In order to determine the amplitude A we integrate the wave equation (58) over a compact sphere of radius a_0 around $\boldsymbol{y}$. Making use of the properties of the delta function we find:

$$-\frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{2\pi} = \int_V (k^2\hat{G}_0 + \nabla^2\hat{G}_0)\,\mathrm{d}V \simeq \int_V \nabla^2\hat{G}_0\,\mathrm{d}V = \int_S \frac{\partial\hat{G}_0}{\partial r}\,\mathrm{d}S = 4\pi a_0^2\Big(\frac{\partial\hat{G}_0}{\partial r}\Big)_{r=a_0}. \tag{111}$$

Using the near field approximation $(\partial\hat{G}_0/\partial r)_{r=a_0} \simeq -A/a_0^2$ we can calculate the amplitude A and we find

$$\hat{G}_0 = \frac{1}{8\pi^2 r}\,\mathrm{e}^{-\mathrm{i}\omega\tau - \mathrm{i}r/c_0} \tag{112}$$

which leads by (generalised) inverse Fourier transformation to

$$G_0 = \frac{1}{4\pi r}\delta(t - \tau - r/c_0). \tag{113}$$

We observe at time t at a distance r from the source a pulse corresponding to the impulsion delivered at the emission time

$$t_e = t - \frac{r}{c_0}. \tag{114}$$

As G_0 depends only on $r = \|\boldsymbol{x} - \boldsymbol{y}\|$ rather than on the individual values of $\boldsymbol{x}$ and $\boldsymbol{y}$, the free space Green's function does not only satisfy the reciprocity relation (65) but also the symmetry relation:

$$\frac{\partial G_0}{\partial x_i} = \frac{\partial G_0}{\partial r}\frac{\partial r}{\partial x_i} = -\frac{\partial G_0}{\partial r}\frac{\partial r}{\partial y_i} = -\frac{\partial G_0}{\partial y_i}. \tag{115}$$

Approaching the source by the listener has the same effect as approaching the listener by the source $\partial r/\partial x_i = -\partial r/\partial y_i$.

3.8 Multipole expansion

We can use the free space Green's function G_0 to obtain a more formal definition of monopoles, dipoles, quadrupoles, etc. As we will see, this corresponds to the use of a Taylor expansion of the free space Green's function. We will consider the *far field* p' in free space of a *compact source* distribution $q(\boldsymbol{x}, t)$. In order to derive the general multipole expansion we will first consider the field at a single frequency. By using the free-field Green's function

$$\hat{G}_0(\boldsymbol{x}|\boldsymbol{y}) = \frac{\mathrm{e}^{-\mathrm{i}kr}}{4\pi r}$$

we find the acoustic field for a given time-harmonic source distribution $\hat{q}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega t}$ in a finite volume V to be given by

$$\hat{p}' = \int_V \hat{q}(\boldsymbol{y})\hat{G}_0(\boldsymbol{x}|\boldsymbol{y})\,\mathrm{d}V_y = \int_V \hat{q}(\boldsymbol{y})\frac{\mathrm{e}^{-\mathrm{i}kr}}{4\pi r}\,\mathrm{d}V_y \tag{116}$$

Suppose the origin is chosen inside V. We are interested in the far field, i.e. $\|\boldsymbol{x}\|$ is large, and a compact source, i.e. kL is small where L is the typical diameter of V. This double limit can be taken in several ways. As we are interested in the radiation properties of the source, which corresponds with $k\|\boldsymbol{x}\| \geq O(1)$, we will keep $k\boldsymbol{x}$ fixed. In that case the limit of small k is the same as small $\boldsymbol{y}$, and we can expand in a Taylor series around $\boldsymbol{y} = \boldsymbol{0}$

$$\begin{aligned} r &= \left(\|\boldsymbol{x}\|^2 - 2(\boldsymbol{x}\cdot\boldsymbol{y}) + \|\boldsymbol{y}\|^2\right)^{1/2} = \|\boldsymbol{x}\|\Big(1 - \tfrac{\boldsymbol{x}\cdot\boldsymbol{y}}{\|\boldsymbol{x}\|^2} + \tfrac{\|\boldsymbol{y}\|^2}{2\|\boldsymbol{x}\|^2} - \tfrac{(\boldsymbol{x}\cdot\boldsymbol{y})^2}{2\|\boldsymbol{x}\|^4} + \ldots\Big) \\ &= \|\boldsymbol{x}\|\Big(1 - \tfrac{\|\boldsymbol{y}\|}{\|\boldsymbol{x}\|}\cos\theta + \tfrac{1}{2}\tfrac{\|\boldsymbol{y}\|^2}{\|\boldsymbol{x}\|^2}\sin^2\theta + \ldots\Big) \end{aligned}$$

where θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$, and

$$\begin{aligned} \frac{\mathrm{e}^{-\mathrm{i}kr}}{r} &= \frac{\mathrm{e}^{-\mathrm{i}k\|\boldsymbol{x}\|}}{\|\boldsymbol{x}\|}\Big(1 + \big(1 + \mathrm{i}k\|\boldsymbol{x}\|\big)\frac{1}{\|\boldsymbol{x}\|^2}\sum_{j=1}^{3} x_j y_j + \ldots\Big) \\ &= \sum_{l,m,n=0}^{\infty} \frac{y_1^l y_2^m y_3^n}{l!\,m!\,n!}\left[\frac{\partial^{l+m+n}}{\partial y_1^l \partial y_2^m \partial y_3^n}\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\right]_{y_1=y_2=y_3=0}. \end{aligned} \tag{117}$$

Utilising the symmetry of r as a function of $\boldsymbol{x}$ and $\boldsymbol{y}$, this is equivalent to

$$\frac{\mathrm{e}^{-\mathrm{i}kr}}{r} = \sum_{l,m,n=0}^{\infty} \frac{(-1)^{l+m+n}}{l!\,m!\,n!} y_1^l y_2^m y_3^n \frac{\partial^{l+m+n}}{\partial x_1^l \partial x_2^m \partial x_3^n}\left[\frac{\mathrm{e}^{-\mathrm{i}k\|\boldsymbol{x}\|}}{\|\boldsymbol{x}\|}\right]. \tag{118}$$

The acoustic field is then given by

$$\hat{p}' = \frac{1}{4\pi}\sum_{l,m,n=0}^{\infty} \frac{(-1)^{l+m+n}}{l!\,m!\,n!}\int_V y_1^l y_2^m y_3^n\,\hat{q}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\frac{\partial^{l+m+n}}{\partial x_1^l \partial x_2^m \partial x_3^n}\left[\frac{\mathrm{e}^{-\mathrm{i}k\|\boldsymbol{x}\|}}{\|\boldsymbol{x}\|}\right]. \tag{119}$$

As each term in the expansion is by itself a solution of the reduced wave equation, this series yields a representation in which the source is replaced by a sum of elementary sources (monopole, dipoles, quadrupoles, in other words, multipoles) placed at the origin ($\boldsymbol{y} = 0$). Expression (119) is the *multipole expansion* of a field from a finite source in Fourier domain. From this result we can obtain the corresponding expansion in time domain.

From the integral formulation (64) we have the acoustic field from a source $q(\boldsymbol{x}, t)$

$$p' = \int_{-\infty}^{\infty} \int_V q(\boldsymbol{y}, \tau) \frac{\delta(t - \tau - r/c_0)}{4\pi r} \, \mathrm{d}\boldsymbol{y} \mathrm{d}\tau = \int_V \frac{q(\boldsymbol{y}, t - r/c_0)}{4\pi r} \, \mathrm{d}\boldsymbol{y} \tag{120}$$

If the dominating frequencies in the spectrum of $q(\boldsymbol{x}, t)$ are low, such that $\omega L/c_0$ is small, we obtain by Fourier synthesis of (119) the multipole expansion in time domain (see Goldstein [35])

$$\begin{aligned} p' &= \frac{1}{4\pi} \sum_{l,m,n=0}^{\infty} \frac{(-1)^{l+m+n}}{l!\,m!\,n!} \frac{\partial^{l+m+n}}{\partial x_1^l \partial x_2^m \partial x_3^n} \left[\frac{1}{\|\boldsymbol{x}\|} \int_V y_1^l y_2^m y_3^n \, q(\boldsymbol{y}, t_e) \, \mathrm{d}\boldsymbol{y} \right] \\ &= \sum_{l,m,n=0}^{\infty} \frac{\partial^{l+m+n}}{\partial x_1^l \partial x_2^m \partial x_3^n} \left[\frac{(-1)^{l+m+n}}{4\pi \|\boldsymbol{x}\|} \mu_{lmn}(t_e) \right] \end{aligned} \tag{121}$$

where $t_e = t - \|\boldsymbol{x}\|/c_0$ is the emission time and $\mu_{lmn}(t)$ is defined by:

$$\mu_{lmn}(t) = \int_V \frac{y_1^l y_2^m y_3^n}{l!\,m!\,n!} q(\boldsymbol{y}, t) \, \mathrm{d}\boldsymbol{y}. \tag{122}$$

The (lmn)-th term of the expansion (121) is called a multipole of order 2^{l+m+n}. The 2^0-order term corresponds to a monopole, a concentrated volume source at $\boldsymbol{y} = 0$ with source strength $\mu_{000} = \int_V q(\boldsymbol{y}, t) \mathrm{d}V_y$, which is called the monopole strength.

Since each term is a function of $\|\boldsymbol{x}\|$ only, the partial derivatives to x_i can be rewritten into expressions containing derivatives to $\|\boldsymbol{x}\|$. In general, these expressions are rather complicated, so we will not try to give the general formulas here.

For very large $\|\boldsymbol{x}\|$ each multipole further simplifies because

$$\frac{\partial}{\partial x_l}\Big(\frac{1}{\|\boldsymbol{x}\|}\mu(t_e)\Big) = \Big(-\frac{\mu'(t_e)}{c_0\|\boldsymbol{x}\|} - \frac{\mu(t_e)}{\|\boldsymbol{x}\|^2}\Big)\frac{x_l}{\|\boldsymbol{x}\|} \simeq -\frac{\mu'(t_e)}{c_0\|\boldsymbol{x}\|}\frac{x_l}{\|\boldsymbol{x}\|} = -\frac{x_l}{c_0\|\boldsymbol{x}\|^2}\frac{\partial}{\partial t}\mu(t_e). \tag{123}$$

This leads to

$$p' \simeq \sum_{l,m,n=0}^{\infty} \frac{x_1^l x_2^m x_3^n}{4\pi (c_0 \|\boldsymbol{x}\|)^{l+m+n} \|\boldsymbol{x}\|} \frac{\partial^{l+m+n}}{\partial t^{l+m+n}} \mu_{lmn}(t_e), \qquad (\|\boldsymbol{x}\| \to \infty). \tag{124}$$

Most results below will be presented in this far-field approximation.

3.9 Doppler effect

We can use the Green's function formalism to determine the effect of the movement of a source on the radiated sound field. We consider a point source localized at the point $\boldsymbol{x}_s(t)$:

$$q(\boldsymbol{x}, t) = Q(t)\delta(\boldsymbol{x} - \boldsymbol{x}_s(t)). \tag{125}$$

For free-field conditions, using equation (113), we find:

$$p'(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \int_V \frac{Q(\tau)\delta(\boldsymbol{y} - \boldsymbol{x}_s(\tau))}{4\pi \|\boldsymbol{y} - \boldsymbol{x}\|} \delta\Big(t - \tau - \frac{\|\boldsymbol{y} - \boldsymbol{x}\|}{c_0}\Big) \, \mathrm{d}V_y \mathrm{d}\tau. \tag{126}$$

After integration over space, using the property (59) of the delta function, we obtain:

$$p'(\boldsymbol{x},t) = \int_{-\infty}^{\infty} \frac{Q(\tau)}{4\pi R}\delta\Big(t-\tau-\frac{R}{c_0}\Big)\mathrm{d}\tau. \tag{127}$$

where

$$\boldsymbol{R}(\tau,\boldsymbol{x}) = \boldsymbol{x}-\boldsymbol{x}_s(\tau), \quad R = \|\boldsymbol{R}\|.$$

The contributions of this integral are limited to the zeros of the argument of the δ-function. In other words, this is an integral of the type

$$\int_{-\infty}^{\infty} F(\tau)\delta(g(\tau))\,\mathrm{d}\tau = \sum_n \int_{t_n-\varepsilon}^{t_n+\varepsilon} F(\tau)\delta((\tau-t_n)\tfrac{\mathrm{d}}{\mathrm{d}\tau}g(t_n))\,\mathrm{d}\tau = \sum_n \frac{F(t_n)}{|\frac{\mathrm{d}}{\mathrm{d}\tau}g(t_n)|} \tag{128}$$

where $\tau = t_n$ correspond to the roots of $g(\tau)$. In the present application we have

$$g(\tau) = t-\tau-\frac{R(\tau,\boldsymbol{x})}{c_0} \tag{129}$$

and so

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = -1+\frac{\boldsymbol{R}\cdot\boldsymbol{v}_s}{Rc_0} = -1+M_r, \quad \text{where } \boldsymbol{v}_s = \frac{\mathrm{d}\boldsymbol{x}_s}{\mathrm{d}\tau}. \tag{130}$$

and M_r is the component of the source velocity $\boldsymbol{v}_s$ in the direction of the listener scaled by the sound speed c_0. We call this the relative Mach number of the source. It is positive for a source approaching the observer and negative for a source receding the observer. It can be shown that for subsonic source velocities $|M_r| < 1$, the equation $g(t_e) = 0$ or

$$c_0(t-t_e) = R(t_e,\boldsymbol{x}) \tag{131}$$

has a single root, which is to be identified as the emission time t_e. Hence, we find for the acoustic field the Liénard-Wiechert potential [54]

$$p'(\boldsymbol{x},t) = \frac{Q(t_e)}{4\pi R(1-M_r)}. \tag{132}$$

When the source moves supersonically along a curve multiple solution t_e can occur. This may lead to a focussing of the sound into certain region of space, leading to the so-called super-bang phenomenon.

The increase (when approaching) or decrease (when receding) of the amplitude is called Doppler amplification, and the factor $(1-M_r)^{-1}$ is called Doppler factor. This Doppler factor is best known from its occurrence in the increase or decrease of pitch of the sound experienced by the listener. For a sound source, harmonically oscillating with frequency ω which is high compared to the typical sound source velocity variations, the listener experiences at time t a frequency

$$\frac{\mathrm{d}(\omega t_e)}{\mathrm{d}t} = \frac{\omega}{1-M_r}. \tag{133}$$

The right-hand side is obtained by implicit differentiation of (131). Hence the observed frequency is the emitted one, multiplied by the Doppler factor.

In this discussion we ignored the physical character of the source. If for example we consider monopole source with a volume injection rate $\Phi_V(t)$ the source is given by:

$$q(\boldsymbol{x},t) = \rho_0 \frac{\partial}{\partial t}\big(\Phi_V(t)\delta(\boldsymbol{x}-\boldsymbol{x}_s(t))\big). \tag{134}$$

The corresponding sound field is:

$$\begin{aligned} p'(\boldsymbol{x},t) &= \frac{\partial}{\partial t}\left[\frac{\rho_0\Phi_V(t_e)}{4\pi R(t_e,\boldsymbol{x})(1-M_r(t_e))}\right] \\ &= \frac{1}{1-M_r(t_e)}\frac{\partial}{\partial t_e}\left[\frac{\rho_0\Phi_V(t_e)}{4\pi R(t_e,\boldsymbol{x})(1-M_r(t_e))}\right]. \end{aligned} \tag{135}$$

Although this is for an arbitrary source Φ_V and path $\boldsymbol{x}_s$ an extremely complex solution, it is interesting to note that even for a constant volume flux Φ_V there is sound production when the source velocity $\boldsymbol{v}_s$ is non-uniform.

In a similar way we may consider the sound field generated by a point force $\boldsymbol{F}(t)$.

$$q(\boldsymbol{x},t) = -\nabla\cdot[\boldsymbol{F}(t)\,\delta(\boldsymbol{x}-\boldsymbol{x}_s(t))]. \tag{136}$$

The produced sound field is given in a far-field approximation by:

$$p'(\boldsymbol{x},t) = \frac{\boldsymbol{R}}{c_0 R(1-M_r)}\cdot\frac{\partial}{\partial t_e}\left[\frac{\boldsymbol{F}}{4\pi R(1-M_r)}\right]. \tag{137}$$

Even when the source flies at constant velocity $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_s'$ this solution involves high powers of the Doppler factor.

An interesting application of this theory is the sound production by a rotating blade. The blade can be represented by a point force (mainly the lift force concentrated in a point) and a compact moving body of constant volume V_b (the blade). The lift noise (the contribution of the lift force) can be calculated by means of equation (137). Note that in practice this lift is not only the steady thrust of the rotating blade but contains also the unsteady component due to interaction of the blades with obstacles like supports or with a turbulent or non-uniform inflow. For the effect of the volume of the blade it can be shown that it is given by the second time derivative

$$p'(\boldsymbol{x},t) = \frac{\partial^2}{\partial t^2}\left[\frac{\rho_0 V_b}{4\pi R(1-M_r)}\right]. \tag{138}$$

Even though the blade volume remains constant, the displacement of air by the rotating blade induces a sound production. This so-called thickness noise depends on a higher power of the Doppler factor than the lift-noise. This implies that at low Mach numbers such as prevails for a ventilation fan, the lift-noise will be dominant. At high source Mach numbers, such as aircraft propellers, the lift noise will still dominate at take off when the required thrust is high, but at cruise conditions the thickness noise becomes comparable, as the tip Mach number is close to unity (or even higher).

We will see in the next section that the sound produced by turbulence in a free jet has the character of the field of a quadrupole distribution. As the vortices that produce the sound are convected with the main flow, there will be a very significant Doppler effect at high Mach numbers. This results into a radiation field mainly directed about the flow direction. Due to convective effects on the wave propagation the sound is, however, deflected in the shear layers of the jet. This explains that along the jet axis there is a so-called cone of silence [49, 35].

3.10 Uniform mean flow, plane waves and edge diffraction

The problem of a source, observer and scattering objects moving together steadily in a uniform stagnant medium is the same as the problem of a fixed source, observer and objects in a uniform mean flow. If the mean flow is in x direction and the perturbations are small and irrotational we have for potential ϕ, pressure p, density ρ and velocity $\boldsymbol{v}$ the problem given by

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c_0^2}\Big(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\Big)^2\phi=0,$$
$$p=-\rho_0\Big(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\Big)\phi,\quad p=c_0^2\rho,\quad \boldsymbol{v}=\nabla\phi \tag{139}$$

where U_0, ρ_0 and c_0 denote the mean flow velocity, density and sound speed, respectively. We assume in the following that $|U_0|<c_0$. The equation for ϕ is known as the convected wave equation.

3.10.1 Lorentz or Prandtl-Glauert transformation

By the following transformation (in aerodynamic context named after Prandtl and Glauert, but qua form originally due to Lorentz)

$$X=\frac{x}{\beta},\quad T=\beta t+\frac{M}{c_0}X,\quad M=\frac{U_0}{c_0},\quad \beta=\sqrt{1-M^2}, \tag{140}$$

the convected wave equation may be associated to a stationary problem with solution $\phi(x,y,z,t)=\psi(X,y,z,T)$ satisfying

$$\frac{\partial^2\psi}{\partial X^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c_0^2}\frac{\partial^2\psi}{\partial T^2}=0,\quad p=-\frac{\rho_0}{\beta}\Big(\frac{\partial}{\partial T}+U_0\frac{\partial}{\partial X}\Big)\psi. \tag{141}$$

For a time-harmonic field $\mathrm{e}^{\mathrm{i}\omega t}\,\phi(x,y,z)=\mathrm{e}^{\mathrm{i}\Omega T}\,\psi(X,y,z)$ or $\phi(x,y,z)=\mathrm{e}^{\mathrm{i}KMX}\,\psi(X,y,z)$, where $\Omega=\omega/\beta$, $k=\omega/c_0$ and $K=k/\beta$, we have

$$\frac{\partial^2\psi}{\partial X^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+K^2\psi=0. \tag{142}$$

The pressure may be obtained from ψ, but since p satisfies the convected wave equation too, we may also associate the pressure field directly by the same transformation with a corresponding stationary pressure field. The results are not equivalent, however, and especially when the field contains singularities some care is in order. The pressure obtained directly is no more singular than the pressure of the stationary problem, but the pressure obtained via the potential is one order more singular due to the convected derivative. See the example below.

3.10.2 Plane waves

A plane wave (in x,y-plane) may be given by

$$\phi_i=\exp\Big(-\mathrm{i}k\frac{x\cos\theta_n+y\sin\theta_n}{1+M\cos\theta_n}\Big)=\exp\Big(-\mathrm{i}kr\frac{\cos(\theta-\theta_n)}{1+M\cos\theta_n}\Big) \tag{143}$$

where θ_n is the direction of the normal to the phase plane and $x = r\cos\theta,\;\; y = r\sin\theta$. This is physically not the most natural form, however, because θ_n is due to the mean flow not the direction of propagation. By comparison with a point source field far away, or from the intensity vector

$$\boldsymbol{I} = (\rho_0\boldsymbol{v} + \rho\boldsymbol{v}_0)(c_0^2\rho/\rho_0 + \boldsymbol{v}_0\cdot\boldsymbol{v}) \sim (\beta^2\tfrac{\partial}{\partial x}\phi - \mathrm{i}kM\phi)\boldsymbol{e}_x + \tfrac{\partial}{\partial y}\phi\boldsymbol{e}_y \sim (M+\cos\theta_n)\boldsymbol{e}_x + \sin\theta_n\boldsymbol{e}_y$$

we can learn that θ_s, the direction of propagation (the direction of any shadows), is given by

$$\cos\theta_s = \frac{M+\cos\theta_n}{\sqrt{1+2M\cos\theta_n+M^2}},\quad \sin\theta_s = \frac{\sin\theta_n}{\sqrt{1+2M\cos\theta_n+M^2}}. \tag{144}$$

By introducing the transformed angle Θ_s

$$\cos\Theta_s = \frac{\cos\theta_s}{\sqrt{1-M^2\sin^2\theta_s}} = \frac{M+\cos\theta_n}{1+M\cos\theta_n}, \tag{145}$$

$$\sin\Theta_s = \frac{\beta\sin\theta_s}{\sqrt{1-M^2\sin^2\theta_s}} = \frac{\beta\sin\theta_n}{1+M\cos\theta_n} \tag{146}$$

and the transformed polar coordinates $X = R\cos\Theta,\;\; y = R\sin\Theta$, we obtain the plane wave

$$\phi_i = \exp\big(\mathrm{i}KMX - \mathrm{i}KR\cos(\Theta-\Theta_s)\big). \tag{147}$$

3.10.3 Half-plane diffraction problem

By using the foregoing transformation, we obtain from the classical Sommerfeld solution for the half-plane diffraction problem (see Jones [56]) of a plane wave (147), incident on a solid half plane along $y = 0,\;\; x < 0$, the following solution (see Rienstra [110])

$$\phi(x,y) = \exp\big(\mathrm{i}KMX - \mathrm{i}KR\big)\big(F(\Gamma_s) + F(\overline{\Gamma}_s)\big) \tag{148}$$

where

$$F(z) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{\pi}}\,\mathrm{e}^{\mathrm{i}z^2}\int_z^\infty \mathrm{e}^{-\mathrm{i}t^2}\,\mathrm{d}t. \tag{149}$$

and

$$\Gamma_s, \overline{\Gamma}_s = (2KR)^{1/2}\sin\tfrac{1}{2}(\Theta\mp\Theta_s). \tag{150}$$

An interesting feature of this solution is the following. When we derive the corresponding pressure

$$p(x,y) = \exp\big(\mathrm{i}KMX - \mathrm{i}KR\big)\big(F(\Gamma_s)+F(\overline{\Gamma}_s)\big) + \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{\pi}}\frac{M\cos\frac{1}{2}\Theta_s}{1-M\cos\Theta_s}\exp\big(\mathrm{i}KMX - \mathrm{i}KR\big)\sin\tfrac{1}{2}\Theta\Big(\frac{2}{KR}\Big)^{1/2}, \tag{151}$$

we see immediately that the first part is just a multiple of the solution of the potential, so the second part has to be a solution too. Furthermore, the first part is regular like ϕ, while this second part is singular at the scattering edge. As the second part decays for any $R\to\infty$, it does not describe the incident plane wave, so it may be dropped if we do not accept the singularity in p at the edge. By considering this solution

$$p_v(x,y) = \exp\big(\mathrm{i}KMX - \mathrm{i}KR\big)\frac{\sin\frac{1}{2}\Theta}{\sqrt{KR}}, \tag{152}$$

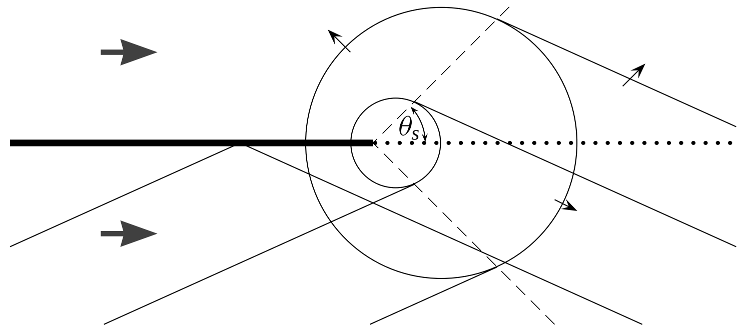

Figure 2: Sketch of scattered plane wave with mean flow

a bit deeper, it transpires that this solution has no continuous potential that decays to zero for large $|y|$ (see [57, 110]). This solution corresponds to the field of vorticity (in the form of a vortex sheet) that is being shed from the edge. This may be more clear if we construct the corresponding potential ϕ_v for large x, which is

$$\phi_v \sim \operatorname{sign}(y)\exp\Big(-\frac{\omega}{U_0}|y| - \mathrm{i}\,\frac{\omega}{U_0}x\Big), \quad p_v \sim 0. \tag{153}$$

In conclusion: we obtain the continuous-potential, singular solution by transforming the no-flow solution in potential form, and the discontinuous-potential, regular solution from the no-flow solution in pressure form. The difference between both is the field of the shed vortex sheet.

The assumption that just as much vorticity is shed that the pressure field is not singular anymore, is known as the unsteady Kutta condition. Physically, the amount of vortex shedding is controlled by the viscous boundary layer thickness compared to the acoustic wave length and the amplitude (and the Mach number for high speed flow). These effects are not included in the present acoustic model, therefore they have to be included by an additional edge condition, for example the Kutta condition. As vorticity can only be shed from a trailing edge, a regular solution is only possible if $M > 0$. If $M < 0$ the edge is a leading edge and we have to leave the singular behaviour as it is.

It may be noted that the same physical phenomenon seems to occur at a transition from a hard to a soft wall in the presence of mean flow. Normally, at the edge there will be a singularity. When the soft wall allows a surface wave of particular type accounting for a modulated vortex sheet along the line surface, a Kutta condition can be applied that removes the singularity, possibly at the expense of a a (spatial) instability [106, 114, 117]. It should be emphasized that this results from a linear model and a too severe instability may be not acceptable in a fully nonlinear model.

4 Aero-acoustic analogies

4.1 Lighthill's analogy

Until now we have considered the acoustic field generated in a quiescent fluid by an imposed external force field $\boldsymbol{f}$ or by heat production Q_w. We have furthermore assumed that the sources induce linear perturbations of the reference quiescent fluid state. Lighthill [68] proposed a generalisation of this approach to the case of an arbitrary source region surrounded by a quiescent fluid. Hence we no more assume that the flow in the source region is a linear perturbation of the reference state. We only assume that the listener is surrounded by a quiescent reference fluid (p_0, ρ_0, s_0, c_0 uniformly constant and $\boldsymbol{v}_0 = 0$) in which the small acoustic perturbations are accurately described by the homogeneous linear wave equation (68). The key idea of Lighthill is to derive from the exact equations of mass conservation (4) with $Q_m = 0$ and momentum conservation (6) a non-homogeneous wave equation which reduces to the homogeneous wave equation (68) in a region surrounding the listener.

By taking the time derivative of the mass conservation law (4) and subtracting the divergence of the momentum equation (6) we obtain:

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j}(\mathcal{P}_{ij} + \rho v_i v_j) - \frac{\partial f_i}{\partial x_i}. \tag{154}$$

By adding the term $c_0^{-2}\frac{\partial^2}{\partial t^2}p'$ to both sides and by making use of definition (7), i.e. $\sigma_{ij} = p\delta_{ij} - \mathcal{P}_{ij}$, we can write (154) as

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j}(\rho v_i v_j - \sigma_{ij}) - \frac{\partial f_i}{\partial x_i} + \frac{\partial^2}{\partial t^2}\Big(\frac{p'}{c_0^2} - \rho'\Big) \tag{155}$$

in which the perturbations p' and ρ' are defined by:

$$p' = p - p_0 \quad \text{and} \quad \rho' = \rho - \rho_0. \tag{156}$$

This equation is called the *analogy* of Lighthill. Note that neither ρ'/ρ_0 nor p'/p_0 are necessarily small in the so-called source region. In fact, equation (155) is exact. Furthermore, as we obtained this equation by adding the term $c_0^{-2}\frac{\partial^2}{\partial t^2}p'$ to both sides, this equation is valid for any value of the velocity c_0. In fact, we could have chosen $c_0 = 3 \times 10^8$ ms^{-1} (the speed of light in vacuum), or $c_0 = 1$ mm/century. Of course, the equation would be quite meaningless then. By choosing for c_0 and p_0 the values of the reference quiescent fluid surrounding the listener, we recover the homogeneous wave equation (68) whenever the right-hand side of equation (155) is negligible. Hence equation (155) is a generalisation of equation (57) which was derived for linear perturbations of a quiescent fluid.

We should now realize that we did not introduce any approximation to equation (155), so this is exact. Therefore, this equation does not provide any new information which was not already contained in the equations of mass conservation (4) and of momentum (6). In fact, we have lost some information. We started with four exact equations (4, 6) and eleven unknowns (v_i, p, ρ, σ_{ij}). We are now left with one equation (155) and still eleven unknowns. Obviously, without additional information and approximations we haven't got any closer to a solution for the acoustic flow.

The first step in making Lighthill's analogy useful was already described above. We have identified a listener around which the flow behaves like linear acoustic perturbations and is described by the homogenous wave equation (68). This is an assumption valid in many applications. When we listen, under normal circumstances, to a flute player we have conditions that are quite reasonably close to these assumptions. At this stage the most important contribution of Lighthill's analogy is that it generalises (156), the equations for the fluctuations ρ' and p' to the entire space, even in a highly nonlinear source region. The next steps will be that we introduce approximations to estimate the source terms, i.e. the right-hand side of equation (155).

We recognize in the right-hand side of (155) the term $\frac{\partial^2}{\partial t^2}(\frac{p'}{c_0^2} - \rho')$ which is a generalisation of the entropy production term in equation (57).

As shown by Morfey [78, 16] this term includes complex effects due to the convection of entropy non-uniformities. The effect of external forces $\boldsymbol{f}$ is the same in both equations (68) and Lighthill's analogy (155). We have, however, now removed the condition that the force should only induce a small perturbation to the reference state. An arbitrary force is allowed, as long as we take into account any additional effects it may have to the other terms at the right-hand side of (155).

We observe additional terms due to the viscous stress σ_{ij} and the Reynolds stress $\rho v_i v_j$. The viscous stress σ_{ij} is induced by molecular transport of momentum while $\rho v_i v_j$ takes into account the nonlinear convection of momentum. One of the key idea of Lighthill [68] is that when the entropy term and the external forces are negligible the flow will only produce sound at high velocities, corresponding to high Reynolds numbers. He therefore assumed that viscous effects are negligible and reduce the sound source to the nonlinear convective effects ($\partial^2 \rho v_i v_j / \partial x_i \partial x_j$). It is worth noting that a confirmation of this assumption being reasonable was provided qualitatively by Morfey [79, 80] and Obermeier [94] only about thirty years after Lighthill's original publication. A quantitative discussion for noise produced by vortex pairing was provided by Verzicco [137]. An additional assumption, commonly used, is that feedback from the acoustic field to the source is negligible. Hence we can calculate the source term from a numerical simulation that ignores any acoustic wave propagation and subsequently predict the sound production outside the flow. In extreme cases of low Mach number flow, a locally incompressible flow simulation of the source region can be used to predict the (essentially compressible!) sound field.

Equation (155) can be formally solved by an integral formulation of the type (64). This will have the additional benefit to reduce the effect of random errors in the source flow on the predicted acoustic field. One can state that such an integral formulation combined with Lighthill's analogy allows to obtain a maximum of information concerning the sound production for a given information about the flow field. A spectacular example of this is Lighthill's prediction that the power radiated to free space by a free turbulent isothermal jet scales as the eight power U_0^8 of the jet velocity. This result is obtained from the formal solution for free space conditions:

$$p'(\boldsymbol{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{-\infty}^{\infty} \int_V \rho v_i v_j \frac{\delta\left(t - \tau - \frac{r}{c_0}\right)}{4\pi r} \mathrm{d}V_y \mathrm{d}\tau = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left[\frac{\rho v_i v_j}{4\pi r}\right]_{\tau = t_e} \mathrm{d}V_y \qquad (157)$$

where $r = \|\boldsymbol{x} - \boldsymbol{y}\|$ and $t_e = t - r/c_0$. Assuming that the sound is produced mainly by the large turbulent structures with a typical length scale of the width D of the jet, we estimate the dominating frequency to be $f = U_0/D$, where U_0 is the jet velocity at the exit of the nozzle. Hence the ratio of jet diameter to acoustical wave length $Df/c_0 = U_0/c_0 = M$. This implies that at low Mach numbers we can neglect variations of the retarded time t_e if the source region is limited to a few pipe diameters. As the acoustic power decreases very quickly with decreasing flow velocity we have indeed a source volume of the order of D^3. As $\boldsymbol{v} \sim U_0$ and $\rho \sim \rho_0$, we may assume that $\rho v_i v_j \sim \rho_0 U_0^2$. From the far field approximation $\partial/\partial x_i = -\partial/(c_0 \partial t) \sim 2\pi f / c_0$ we find:

$$p'(\boldsymbol{x}, t) \sim \rho_0 U_0^2 M^2 \frac{D}{\|\boldsymbol{x}\|}, \qquad (158)$$

where we ignored the effect of convection on the sound production [35, 49]. The radiated power is thus found to be

$$\langle I \rangle \simeq 4\pi \|\boldsymbol{x}\|^2 \langle p' v_r' \rangle \sim \rho_0 U_0^3 M^5 D^2 \sim U_0^8. \qquad (159)$$

This scaling law appears indeed to be quite accurate for subsonic isothermal free jets. Note that this law implies that we can achieve a dramatic reduction of aircraft jet noise production by reducing the flow Mach number. In order to retain the necessary thrust the jet cross section has to be increased. This is is exactly what happened in the 60's and 70's when the high-bypass turbofan aeroengines replaced the older turbofan engines without or with little by-pass flow. This is known as the "turbofan revolution" [130, 49].

As stated by Crighton [16] and Powell [104], Lighthill's theory is rather unique in its predicting a physical phenomenon before experiments were accurate enough to verify it. This made Lighthill's analogy famous. Note that we have actually discarded any contribution from entropy fluctuations or external forces. This means that if we use, as input for the analogy, data obtained from a numerical simulation that includes significant viscous dissipation and spurious forces, we still would predict the same scaling law.

As the amplitude of the acoustic pressure generated by a compact monopole and dipole would scale in free field conditions as respectively M^2 and M^3, these spurious sources easily dominate the predicted sound amplitude. This is one reason for which most of the direct numerical simulations of sound production by a subsonic flow in free field conditions are carried out at high Mach numbers ($M \simeq 0.9$).

In the presence of walls the sound radiation by turbulence can be dramatically enhanced. In the next section, we will see that compact bodies will radiate a dipole sound field associated with the force which they exert on the flow as a reaction to the hydrodynamic force of the flow applied to them. Sharp edges are particularly efficient radiators. This corresponds to our common experience with the production of sounds like the consonant $/s/$. The interaction of the sharp edges of our teeth is essential. In free field conditions much attention has been devoted to the so-called trailing edge noise of aircraft wings [35, 49, 6, 47]. The scaling rule for this sound field is $p' \sim M^{5/2}$, which is just in-between a monopole and a dipole. For confined subsonic flows at low frequencies the scaling rules are quite different. A compact turbulent flow in an infinitely long straight duct will produce an acoustic field, that scales according to $p' \sim M^6$ [32, 73, 119]. The sound field from a monopole and an axially aligned dipole will both scale with M^2, while transversal dipoles will not radiate any sound. We will consider elements of the acoustics and aero-acoustics of confined flows in section (5). The scaling rules for supersonic free jets is obscured by the temperature difference between the flow and the environment [131]. Globally, however, one expects a power that is limited by a U_0^3-law, because an extrapolation of the scaling rule $\sim U_0^8$ would imply that the acoustical power generated by the flow would soon become larger than the kinetic energy flux of the flow, which scales as $\frac{1}{2}\rho_0 U_0^3$. At high Mach numbers, the theory has to be modified in order to take intrinsic temperature differences into account.

4.2 Curle's formulation

The integral formulation of Lighthill's analogy can be generalised for flows in the presence of wall's. We will later discuss the use of tailored Green's functions (section 5.3). We now consider the approach of Curle [17]. We use the free space Green's function G_0 (114). We use instead of the pressure p' as aeroacoustical variable the density ρ'. Subtracting from both sides of equation (154) the term $c_0^2(\partial^2\rho'/\partial x_i^2)$ we obtain the analogy of Lighthill for the density:

$$\frac{\partial^2\rho'}{\partial t^2} - c_0^2\frac{\partial^2\rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} - \frac{\partial f_i}{\partial x_i} \tag{160}$$

in which the stress tensor of Lighthill T_{ij} is defined by:

$$T_{ij} = \mathcal{P}_{ij} + \rho v_i v_j - c_0^2 \rho \delta_{ij}. \tag{161}$$

We further assume that $\boldsymbol{f} = 0$ and focus on the other sound sources. We have selected here the density as the dependent variable, because this was Lighthill's [68] original choice. We will later discuss the implications of this choice (section 4.4).

We consider a fixed surface S with outer normal $\boldsymbol{n}$ and we apply Green's theorem (64) to the volume V outside of S. Note that $\boldsymbol{n}$ is chosen towards the interior of V, so the sign convention of $\boldsymbol{n}$ is opposite to the sign convention used in (64). By means of partial integration and utilising the symmetry properties $\partial G_0/\partial x_i = -\partial G_0/\partial y_i$ (120) and $\partial G_0/\partial \tau = -\partial G_0/\partial t$ of the Green's function G_0, we obtain [35, 119]:

$$p'(\boldsymbol{x}, t) = c_0^2 \rho'(\boldsymbol{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left[\frac{T_{ij}}{4\pi r} \right]_{\tau=t_e} \mathrm{d}V_y + \frac{\partial}{\partial t} \int_S \left[\frac{\rho v_i}{4\pi r} \right]_{\tau=t_e} n_i \, \mathrm{d}S - \frac{\partial}{\partial x_j} \int_S \left[\frac{\mathcal{P}_{ij} + \rho v_i v_j}{4\pi r} \right]_{\tau=t_e} n_i \, \mathrm{d}S \tag{162}$$

which is clearly a generalisation of (157). In this equation we used the assumption that at the listener's position $p' = c_0^2 \rho'$. In the far-field approximation (123) we find:

$$p'(\boldsymbol{x}, t) \simeq \frac{x_i x_j}{4\pi \|\boldsymbol{x}\|^2 c_0^2} \frac{\partial^2}{\partial t^2} \int_V \left[\frac{T_{ij}}{r} \right]_{\tau=t_e} \mathrm{d}V_y + \frac{1}{4\pi} \frac{\partial}{\partial t} \int_S \left[\frac{\rho v_i}{r} \right]_{\tau=t_e} n_i \, \mathrm{d}S + \frac{x_j}{4\pi \|\boldsymbol{x}\| c_0} \frac{\partial}{\partial t} \int_S \left[\frac{\mathcal{P}_{ij} + \rho v_i v_j}{r} \right]_{\tau=t_e} n_i \, \mathrm{d}S. \tag{163}$$

For a compact body we can neglect the variations of t_e over the surface and we can write $r = \|\boldsymbol{x}\|$ if we chose the origin $\boldsymbol{y} = 0$ inside or near the body. Assuming that T_{ij} decays fast enough, we have in that case:

$$p'(\boldsymbol{x}, t) \simeq \frac{x_i x_j}{4\pi \|\boldsymbol{x}\|^3 c_0^2} \frac{\partial^2}{\partial t^2} \int_V \left[T_{ij} \right]_{\tau=t_e} \mathrm{d}V_y + \frac{1}{4\pi \|\boldsymbol{x}\|} \frac{\partial}{\partial t} \int_S \left[\rho v_i \right]_{\tau=t_e} n_i \, \mathrm{d}S + \frac{x_j}{4\pi \|\boldsymbol{x}\|^2} \frac{\partial}{\partial t} \int_S \left[\mathcal{P}_{ij} + \rho v_i v_j \right]_{\tau=t_e} n_i \, \mathrm{d}S. \tag{164}$$

and $t_e = t - \|\boldsymbol{x}\|/c_0$. The second integral corresponds to the monopole sound field generated by the mass-flux through the surface S. The third integral corresponds to the dipole sound field generated by the instantaneous force $-F_j$ of the surface to the surrounding fluid. This is the reaction of the surface to the force $F_j = -\int_S (\mathcal{P}_{ij} + \rho v_i v_j) n_i \, \mathrm{d}S$of the flow on the surface. This result is a generalisation of Gutin's principle [36, 35].

Using this theory we understand easily that a rotor blade moving in a non-uniform flow field will generate sound due to the unsteady hydrodynamic forces on the blade. At low Mach number this will easily dominate the Doppler effect due to the rotation. Wind rotors placed downwind of the supporting mast are cheap because they are hydrodynamically stable. There is no need for a feedback system to keep them in the wind. However, the interaction of the wake of the mast with the rotor blades causes dramatic noise problems [50, 138].

4.3 Ffowcs Williams-Hawkings formulation

While Curle's formulation discussed in the previous section assumes a fixed control surface S, the formulation of Ffowcs Williams and Hawkings allows the use of a moving control surface $S(t)$. The key idea is to include the effect of the surface in the differential equation (160) [33, 35, 16]. This is achieved by a most elegant and efficient utilisation of generalised functions (viz. so-called surface distributions).

We assume that the volume $B(t)$ enclosed by the surface $S(t)$ and this surface are sufficiently smooth to allow the definition of a smooth function $h(\boldsymbol{x}, t)$ such that

$$\left.\begin{array}{ll} h(\boldsymbol{x}, t) < 0 & \text{if } \boldsymbol{x} \in B(t) \\ h(\boldsymbol{x}, t) = 0 & \text{if } \boldsymbol{x} \in S(t) \\ h(\boldsymbol{x}, t) > 0 & \text{outside } B(t). \end{array}\right\} \tag{165}$$

Consider any physical quantity, like ρ', defined outside $B(t)$, and extend its definition to all space by giving it a value equal to zero inside $B(t)$. This is efficiently done by assuming ρ' smoothly defined everywhere. Then by multiplying it by the Heaviside function $H(h)$ we create a new variable $\rho' H(h)$ which vanishes within the volume $B(t)$ (where $H(h) \equiv 0$) and is equal to ρ' outside $B(t)$ (where $H(h) \equiv 1$). The next step will be to extend the prevailing equations to the whole space by adding suitable surface sources. To achieve this we need the normal $\boldsymbol{n}$ to the surface $S(t)$, given by

$$\boldsymbol{n} = \left[\frac{\nabla h}{\|\nabla h\|} \right]_{h=0}. \tag{166}$$

We assume that the surface $S(t)$ is parameterized[2] in time and space by the coordinates $(t; \lambda, \mu)$. A point $\boldsymbol{x}_s(t) \in S(t)$ with fixed parameters λ and μ is moving with the velocity $\boldsymbol{b}$. Hence we have $h(\boldsymbol{x}_s, t) = 0$ and

$$\frac{\partial h}{\partial t} = -\boldsymbol{b} \cdot \nabla h = -(\boldsymbol{b} \cdot \boldsymbol{n}) \|\nabla h\|. \tag{167}$$

After multiplying the mass conservation equation (4) and the momentum equation (6) by $H(h)$ and reordering terms, we obtain the following equations valid everywhere

$$\frac{\partial}{\partial t}[\rho' H] + \nabla \cdot [\rho \boldsymbol{v} H] = [\rho_0 \boldsymbol{b} + \rho(\boldsymbol{v} - \boldsymbol{b})] \cdot \nabla H, \tag{168a}$$

$$\frac{\partial}{\partial t}[\rho \boldsymbol{v} H] + \nabla \cdot \big[(\boldsymbol{\mathcal{P}} + \rho \boldsymbol{v}\boldsymbol{v}) H\big] = [\boldsymbol{\mathcal{P}} + \rho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{b})] \cdot \nabla H, \tag{168b}$$

where H stands for $H(h)$. As $\nabla H = \delta(h) \nabla h$, the equations can be interpreted as generalisations of the mass and momentum equations with surface sources at S. Using the above relations and following Lighthill's procedure for acoustic variable $p' = p - p_0$, we find:

$$\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2}[p' H] - \nabla^2 [p' H] = \nabla \cdot \big[\nabla \cdot [(\rho \boldsymbol{v}\boldsymbol{v} - \boldsymbol{\sigma}) H] \big] - \nabla \cdot [\boldsymbol{f} H] + \frac{\partial^2}{\partial t^2} \left[\Big(\frac{p'}{c_0^2} - \rho' \Big) H \right] + \frac{\partial}{\partial t} \big[\big(\rho_0 \boldsymbol{b} + \rho(\boldsymbol{v} - \boldsymbol{b}) \big) \cdot \nabla H \big] - \nabla \cdot \big[\big(p' \boldsymbol{\mathcal{I}} - \boldsymbol{\sigma} + \rho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{b}) \big) \cdot \nabla H \big], \tag{169}$$

[2] When $S(t)$ is the surface of a solid and undeformable body, it is natural to assume a spatial parametrisation which is materially attached to the surface. Like the auxiliary function h, this parametrisation is not unique.

where $(\mathcal{I})_{ij} = \delta_{ij}$ and p_0 is the uniform reference value of the pressure. Note that $\nabla\cdot(\nabla\cdot(p_0\mathcal{I}H)) = \nabla\cdot(p_0\mathcal{I}\cdot\nabla H)$. For a solid surface $\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{b}\cdot\boldsymbol{n}$. In that case by applying Green's theorem and using the free-space Green's function we find:

$$
\begin{aligned}
p'(\boldsymbol{x},t) = {} & \frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbb{R}^3}\left[\frac{(\rho v_i v_j - \sigma_{ij})H}{4\pi r}\right]_{\tau=t_e}\mathrm{d}V_y - \frac{\partial}{\partial x_i}\int_{\mathbb{R}^3}\left[\frac{\boldsymbol{f}H}{4\pi r}\right]_{\tau=t_e}\mathrm{d}V_y \\
& + \frac{\partial^2}{\partial t^2}\int_{\mathbb{R}^3}\left[\frac{(p'/c_0^2 - \rho')H}{4\pi r}\right]_{\tau=t_e}\mathrm{d}V_y + \frac{\partial}{\partial t}\int_{S(t_e)}\left[\frac{\rho_0\,\boldsymbol{b}\cdot\boldsymbol{n}}{4\pi r(1-M_r)}\right]_{\tau=t_e}\mathrm{d}S \\
& - \frac{\partial}{\partial x_i}\int_{S(t_e)}\left[\frac{p' n_i - \sigma_{ij} n_j}{4\pi r(1-M_r)}\right]_{\tau=t_e}\mathrm{d}S, \qquad (170)
\end{aligned}
$$

where $r = \|\boldsymbol{x}-\boldsymbol{y}\|$ and $M_r = \boldsymbol{b}\cdot(\boldsymbol{x}-\boldsymbol{y})/rc_0$ and we used the following generalisations of equation (128)

$$\int_{\mathbb{R}^3} g(\boldsymbol{x})\delta(h(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} = \int_S \frac{g(\boldsymbol{x})}{|\nabla h|}\,\mathrm{d}S, \qquad (171a)$$

$$\int_{\mathbb{R}^3} \boldsymbol{g}(\boldsymbol{x})\cdot\nabla H(h(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3}(\boldsymbol{g}\cdot\nabla h)\delta(h)\,\mathrm{d}\boldsymbol{x} = \int_S \frac{\boldsymbol{g}\cdot\nabla h}{|\nabla h|}\,\mathrm{d}S = \int_S \boldsymbol{g}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S. \qquad (171b)$$

The first three integrals correspond to the contribution of the flow around the surface $S(t)$ while the last two integrals represent generalisations of the thickness noise and sound generated by the surface forces which we have discussed earlier. A reduced form, widely used for subsonic propeller and fan noise when volume sources and surface stresses are negligible, is thus [31]

$$p'(\boldsymbol{x},t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\int_{S(t_e)}\left[\frac{\rho_0\,\boldsymbol{b}\cdot\boldsymbol{n}}{r(1-M_r)}\right]_{\tau=t_e}\mathrm{d}S - \frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{S(t_e)}\left[\frac{p' n_i}{r(1-M_r)}\right]_{\tau=t_e}\mathrm{d}S. \qquad (172)$$

4.4 Choice of aero-acoustic variable

In the previous discussion we used p' as the dependent aero-acoustical variable to introduce the analogy of Lighthill (section 4.1). We then used ρ' to introduce the formulation of Curle describing the effects of stationary boundaries (section 4.2). Finally, for the formulation of Ffowcs Williams and Hawking, describing the effect of moving boundaries, we retuned to p'. In acoustics this would have been indifferent as the two variables are related by the equation of state $p' = c_0^2\rho'$. As we assume linear acoustics of quiescent fluids to be valid around the listener, we made use of this relationship in equation (162). Actually, in aero-acoustics there is a subtle difference which appears when we compare the source terms of the two wave equations:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j}(\rho v_i v_j - \sigma_{ij}) - \frac{\partial f_i}{\partial x_i} + \frac{\partial^2}{\partial t^2}\left(\frac{p'}{c_0^2} - \rho'\right) \qquad (155)$$

and:

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2\frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j}(\rho v_i v_j - \sigma_{ij}) - \frac{\partial f_i}{\partial x_i} + \frac{\partial^2}{\partial x_i^2}(p' - c_0^2\rho'). \qquad (173)$$

Without further approximation, these two forms are both equivalent. However, considered as an analogy the right-hand sides are assumed to be known and act as given source distribution. In that case we see that when p' is used as the aero-acoustical variable the effect of entropy fluctuations

$\partial^2((p'/c_0^2) - \rho')/\partial t^2$ has the character of a monopole sound source. On the other hand, when ρ' is used the apparently *same* effects produce a quadrupole distribution $\partial^2(p' - c_0^2\rho')/\partial x_i^2$ which is qualitatively different. Of course, there is no difference if we consider the exact equations, but if we do not introduce any approximation the analogy is just a reformulation of basic equations without much use. Clearly, we have to be careful in selecting the aero-acoustic variable.

When considering sound production by subsonic flames we should choose p' as aero-acoustical variable because most of the sound is produced by the volume changes associated with the combustion. When we neglect convection effects we have exactly the source term $\partial^2((p'/c_0^2) - \rho')/\partial t^2$. As shown by Morfey [78, 16] this term includes complex effects due to the convection of entropy non-uniformities. They become explicit when we use the equation of state (15) as applied to a material element:

$$\frac{\mathrm{D}p}{\mathrm{D}t} = c^2\frac{\mathrm{D}\rho}{\mathrm{D}t} + \left(\frac{\partial p}{\partial s}\right)_\rho \frac{\mathrm{D}s}{\mathrm{D}t} \tag{174}$$

in combination with (12):

$$T\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{\mathrm{D}e}{\mathrm{D}t} + p\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right). \tag{175}$$

Morfey [78] obtains the following result:

$$-\frac{\partial^2\rho_e}{\partial t^2} = \frac{\partial}{\partial t}\left[\left(\frac{c^2}{c_0^2} - 1 + \frac{\rho_e}{\rho}\right)\frac{\mathrm{D}\rho'}{\mathrm{D}t} + \frac{\rho^2}{c_0^2}\left(\frac{\partial T}{\partial \rho}\right)_s \frac{\mathrm{D}s'}{\mathrm{D}t} + \nabla\cdot(\rho_e \boldsymbol{v})\right] \tag{176}$$

where the excess density ρ_e is defined by $\rho_e = \rho' - (p'/c_0^2)$. The first term vanishes in a subsonic free jet of an ideal gas with constant heat capacity. The second term is the entropy production (combustion) term corrected for convective effects. The last term corresponds to the force exerted by a patch of fluid with a different density on it surroundings in an accelerating flow. This may be compared with a buoyancy (Archimedes) effect. It is induced by the fact that due to the difference of density between the particle and its surroundings, the pressure gradient imposed by the surroundings of a particle does not match its acceleration.

The question arises whether ρ' could be a useful choice too. If we consider a turbulent isentropic flow in a region with a speed of sound c which differs strongly from the speed of sound c_0 at the listener position, the source term can be rewritten as $\partial^2 p'(1 - (c_0/c)^2)/\partial x_i^2$. We expect in a subsonic turbulent flow the local pressure fluctuations p' to scale with $\frac{1}{2}\rho U_0^2$. Hence the analogy indicates that when the speed of sound c_0 at the listener position is much higher than the speed of sound c in the source region, there will be a strong enhancement of the sound production compared to a similar flow in a uniform fluid. Such a spectacular effect does indeed occur when we consider a flow in a water-air mixture such as obtained by directing the nozzle of the shower towards the surface of the water. Due to air entrainment there will be a volume fraction β of air in the water flow. The density of the mixture will be:

$$\rho = (1-\beta)\rho_{\mathrm{water}} + \beta\rho_{\mathrm{air}}. \tag{177}$$

If we assume a quasi-static response of the bubbles and neglect dissolution of air in the water, the compressibility $1/(\rho c^2)$ of the mixture will be the sum of the compressibilities of both phases [141, 16]:

$$\frac{1}{\rho c^2} = \frac{1-\beta}{\rho_{\mathrm{water}}c_{\mathrm{water}}^2} + \frac{\beta}{\rho_{\mathrm{air}}c_{\mathrm{air}}^2}. \tag{178}$$

In the case of water-air mixtures for not too small nor too large values of β the density is mainly determined by the water phase, while the air determines the compressibility. Hence we find for the speed of sound:

$$c^2 \simeq \frac{\rho_{\mathrm{air}} c_{\mathrm{air}}^2}{\beta(1-\beta)\rho_{\mathrm{water}}}. \tag{179}$$

Typical values are $\beta = 0.5$, $\rho_{\mathrm{air}} c_{\mathrm{air}}^2 = 1.4 \times 10^5$ Pa , $\rho_{water} = 10^3$kg m^{-3} and $c_{\mathrm{water}} = 1.5 \times 10^3$ ms^{-1}. We find $c \simeq 2 \times 10^1$ ms^{-1} and if we have our head underwater $(c_0/c)^2 \simeq 5 \times 10^3$. This should result into an enhancement in SPL of the order of 60 dB. Indeed the flow is much noisier than when we avoid air entrainment by putting the shower nozzle underwater.

Other choices of aero-acoustic variables lead to different analogies. In many cases such analogies tend to avoid the problem induced by the fact that the analogy of Lighthill does not distinguish between propagation and production of sound waves in a strongly non-uniform flow, which induces refraction. This becomes very important in supersonic flow. In such cases the source is not compact. Simple results like equation (161) are not valid any more. Alternative analogies which attempt to overcome such problems are described in the literature [35, 49, 70].

4.5 Vortex Sound

One of the drawbacks of the analogy of Lighthill (155) is that the source term is spatially quite extended. As observed by Powell [103] the sound production in subsonic homentropic flows is associated to the dynamics of vortices. Vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ appears to be spatially less extended than the Reynolds stress $\rho v_i v_j$. The reason for this is that around vortices there is a large region of potential flow which actually does not produce any sound. In its original form the Vortex Sound Theory [103] was applied to free-field conditions at low Mach numbers. It was only a special form of Lighthill's analogy stressing the role of vorticity. In the case of free field conditions the Vortex Sound Theory enhances very much the predicted sound field. Various modifications of the theory of Powell have been proposed which impose more explicitly the conservation of momentum and energy to the flow in the source region [75, 41, 127]. This also improves the performance of the theory.

Howe [44, 47, 48] has generalised the theory of Powell [103] to allow its application to confined flows and conditions in which the listener is placed into a potential flow rather than a quiescent fluid. In this theory the fluctuations of the total enthalpy $B' = (p'/\rho_0) + v'v_0$ appears as a natural aero-acoustical variable [44, 23, 24, 86]. In its general form the Vortex Sound Theory can be applied to arbitrary Mach numbers. A similar analogy was derived by Möhring [76] on the basis of acoustical energy considerations. However, such analogies become quite obscure. They do not provide much intuitive insights. They can only be used numerically as proposed by Schröder [30].

We consider now the case of low subsonic flows in which Howe [46] proposed a very nice energy corollary which provides much insight into the role of vorticity in sound production. We propose here an intuitive approach to this theory. The key idea of the theory is that a potential flow is silent. This is illustrated in figure 3 in which we consider a sketch of our vocal folds. The oscillation of the vocal folds result into a variable volume flow from our lungs into the vocal tract. This variable volume flux is the source of sound. Seeking a simplified model for the flow through the glottis, we consider the Reynolds number Re and the Strouhal number St. For a typical flow velocity $U_0 = 30\,\mathrm{ms}^{-1}$, a length scale of the glottis of the order of $L = 2\,\mathrm{mm}$, a frequency $f = 10^2$ Hz and a kinematic viscosity $\nu = 1.5 \times 10^{-5}$ m^{2}s^{-1} we have: $Re = U_0 L/\nu = \mathcal{O}(10^3)$ and $St = fL/U_0 = \mathcal{O}(10^{-2})$. A quasi-steady frictionless approximation seems promising. As the Mach number is low $M = \mathcal{O}(10^{-1})$ and the flow is compact $L/\lambda = MSt = \mathcal{O}(10^{-3})$ we assume that the flow is locally incompressible. Under such circumstances the equation of Bernoulli in the form $p + \frac{1}{2}\rho_0 v^2 = p_t$ with p_t a constant, should be valid. About one diameter upstream and downstream of the vocal folds we expect a uniform flow velocity. Assuming the flow channel upstream of the vocal folds (trachea) to have the same cross section as the channel downstream (vocal tract) we find that the velocities should be equal. As a consequence we conclude that there is no pressure difference across the vocal folds. The volume flux is therefore not controlled by the opening of the folds! This implies that they cannot produce any sound. This corresponds to the paradox of d'Alembert. This paradox is solved by realizing that we can never neglect the effect of friction at the walls. Even at high Reynolds numbers there are always thin viscous boundary layers near the walls. In these boundary layers the pressure is essentially equal to that of the main flow but the velocity decreases to zero at the wall. This implies that the stagnation pressure p_t is lower than its value in the main flow. Hence, the fluid in the boundary layers cannot flow against the strong adverse pressure gradient in the diverging part of the glottis. This results into separation of the boundary layers and the formation of a free jet. Turbulent dissipation in the free jet explains the flow control by the oscillating vocal folds and the pressure difference across the glottis. Of course, the fluid from the boundary layers injected into the main flow has vorticity. Hence we see that vorticity injection into the main flow is associated with the production of sound. This illustrates the statement of Müller and Obermeier [83]: "Vortices are the voice of the flow".

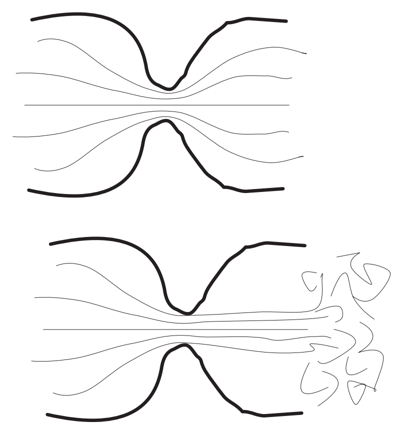

Figure 3: A potential flow through the vocal folds is silent. Sound is produced by the volume flow control associated to flow separation and formation of a free jet. This implies vorticity injection in the main flow.

As explained in section (2.3) the velocity field can be separated into a potential and a vortical flow. The potential part $\nabla\phi$ of the flow is associated with the dilatation rate $\nabla\cdot\boldsymbol{v} = \nabla^2\phi$ of fluid particles in the flow. As the acoustical flow is essentially compressible and unsteady, Howe [46] proposed to define the acoustic field as the unsteady component of the potential flow $\boldsymbol{u}_{ac}$:

$$\boldsymbol{u}_{ac} = \nabla\phi', \tag{180}$$

in which $\phi' = \phi - \phi_0$ is the time dependent part of the potential ϕ. For an homentropic flow we can write the equation of Euler (33) in the form:

$$\frac{\partial\boldsymbol{v}}{\partial t} + \nabla B = \frac{\boldsymbol{f}_c}{\rho} \tag{181}$$

in which $\boldsymbol{f}_c = -\rho(\boldsymbol{\omega}\times\boldsymbol{v})$ is the density of the Corriolis force associated with the vorticity of the flow. When $\boldsymbol{\omega} = 0$ we have a potential flow. Hence we identify $\boldsymbol{f}_c/\rho$ as the source of sound. At low Mach number for compact flows we can neglect the variation of the density so that $\boldsymbol{f}_c$ is the source of sound. Under such circumstances we can in first approximation apply the energy equation (105) in the form:

$$\langle P\rangle = \int_V \langle \boldsymbol{f}_c\cdot\boldsymbol{u}_{ac}\rangle\, \mathrm{d}V. \tag{182}$$

where $\langle P\rangle$ is the time average acoustical power generated by the vortices and V is the volume in which the vorticity $\boldsymbol{\omega}$ is confined.

The success of this equation is due to the fact that even when using a highly simplified vortex model it provides a fair prediction of the sound production. A first explanation for this is that it is an integral formulation which smoothes out random errors. A second aspect is that when considering models of isolated vortices with time dependent circulation Γ it ignores the effect of the spurious pressure difference $-\rho\frac{\partial}{\partial t}\Gamma$ induced by the model along the "feeding line" of the vortex. This corresponds to the force necessary for the transport of vorticity towards the "free" vortex. Hence a model which does not satisfy the momentum equation still gives reasonable predictions because we ignore the effect of spurious forces.

The theory of Howe provides much insight. In particular, equation (182) clarifies the essential role of sharp edges at which the potential flow is singular and where vortex shedding occurs. In section (5.4) we will consider examples of the use of this theory.

For the sake of completeness we provide now a more formal form of the analogy of Howe [48]. Starting from the divergence of the momentum conservation equation (181):

$$\frac{\partial \nabla\cdot\boldsymbol{v}}{\partial t} + \nabla^2 B = -\nabla\cdot(\boldsymbol{\omega}\times\boldsymbol{v}) \tag{183}$$

and eliminating $\nabla\cdot\boldsymbol{v}$ by means of the mass conservation law (3):

$$\nabla\cdot\boldsymbol{v} = -\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} \tag{184}$$

we obtain for an isentropic flow:

$$\frac{1}{c^2}\frac{\mathrm{D}_0^2 B'}{\mathrm{D}t^2} - \nabla^2 B' = \nabla\cdot(\boldsymbol{\omega}\times\boldsymbol{v}) + \frac{1}{c^2}\frac{\mathrm{D}_0^2 B'}{\mathrm{D}t^2} - \frac{\partial}{\partial t}\frac{1}{c^2}\frac{\mathrm{D}i}{\mathrm{D}t} \tag{185}$$

where:

$$\frac{\mathrm{D}_0 B'}{\mathrm{D}t} = \frac{\partial B'}{\partial t} + (\nabla\phi_0)\cdot\nabla B' \tag{186}$$

and we made use of the equation of state for isentropic flows:

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c^2\rho}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{1}{c^2}\frac{\mathrm{D}i}{\mathrm{D}t}. \tag{187}$$

Following Howe [47] the first source term $\nabla\cdot(\boldsymbol{\omega}\times\boldsymbol{v})$ in (185) is dominant at low Mach numbers. While the resulting wave equation (185) looks relatively simple, for an arbitrary steady potential flow $\nabla\phi_0$ it can only be solved numerically.

5 Confined flows

5.1 Wave propagation in a duct

For a narrow hard-walled duct the only waves that propagate are of the form given by equation (70), which is for a time-harmonic sound field in complex notation the pair of plane waves

$$p'(\boldsymbol{x},t) = A\,\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}kx} + B\,\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}kx}, \tag{188}$$

where $k = \omega/c_0$ and the medium is uniform and stagnant. Note that each plane wave is self-similar in x apart from a phase change. This solution can be generalised for wider ducts or higher frequencies as follows.

The time-harmonic sound field in a duct of constant cross section with linear boundary conditions that are independent of the axial coordinate may be described by an infinite sum of special solutions – modes – that retain their shape when travelling down the duct. They consist of an exponential term multiplied by an eigenfunction of the Laplace eigenvalue problem on a duct cross section.

Consider the two-dimensional area $\mathcal{A}$ in (y,z)-plane with a smooth boundary $\partial\mathcal{A}$ and an externally directed unit normal $\boldsymbol{n}$. By shifting $\mathcal{A}$ in x-direction we obtain the duct $\mathcal{D}$ given by

$$\mathcal{D} = \{(x,y,z) | (0,y,z) \in \mathcal{A}\} \tag{189}$$

with axial cross sections being copies of $\mathcal{A}$ and where the normal vectors $\boldsymbol{n}$ are the same for all x (figure 4). In the usual complex notation (with $+\mathrm{i}\omega t$–sign convention), the acoustic field

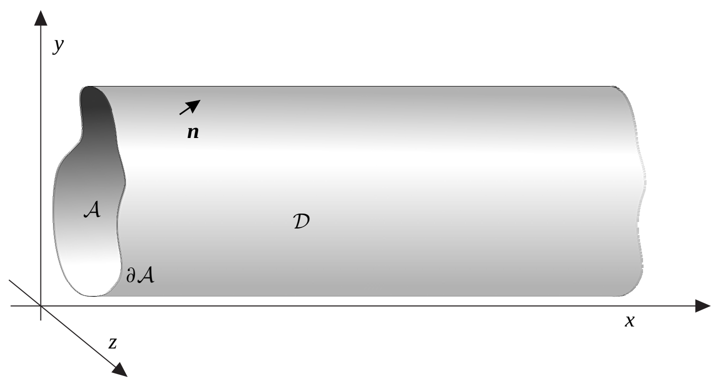

Figure 4: Straight duct of arbitrary cross section

$$p'(\boldsymbol{x},t) \equiv p'(\boldsymbol{x},\omega)\,\mathrm{e}^{\mathrm{i}\omega t}, \quad \boldsymbol{v}'(\boldsymbol{x},t) \equiv \boldsymbol{v}'(\boldsymbol{x},\omega)\,\mathrm{e}^{\mathrm{i}\omega t} \tag{190}$$

satisfies in the duct ($\boldsymbol{x} \in \mathcal{D}$) the equations

$$\nabla^2 p' + k^2 p' = 0, \quad \mathrm{i}\omega\rho_0 \boldsymbol{v}' + \nabla p' = 0. \tag{191}$$

At the duct wall we assume the impedance boundary condition

$$p' = Z(\boldsymbol{v}' \cdot \boldsymbol{n}) \ \text{ for } \ \boldsymbol{x} \in \partial\mathcal{D}. \tag{192}$$

Hard walls correspond with $Z = \infty$. The solution of this problem may be given by

$$p'(x,y,z) = \sum_{n=1}^{\infty} C_n \psi_n(y,z)\,\mathrm{e}^{-\mathrm{i}\kappa_n x} \tag{193}$$

where ψ_n are the eigenfunctions of the Laplace operator reduced to $\mathcal{A}$, i.e. solutions of

$$\begin{aligned} -\Big(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)\psi &= \alpha^2 \psi \ \text{ for } \ (y,z) \in \mathcal{A}, \\ -\mathrm{i}\omega\rho_0 \psi &= (\boldsymbol{n} \cdot \nabla\psi) Z \ \text{ for } \ (y,z) \in \partial\mathcal{A}, \end{aligned} \tag{194}$$

where α^2 is the corresponding eigenvalue. The axial wave number κ_n is given by one of the square roots $\kappa_n = \pm\sqrt{k^2 - \alpha_n^2}$, $+$ for right and $-$ for left running, while the branch of the squareroot is to be taken such that $\mathrm{Re}(\kappa_n) \geq 0$ and $\mathrm{Im}(\kappa_n) \leq 0$. Therefore, the left-running and right-running modes are usually explicitly given as

$$p'(x, y, z) = \sum_{n=1}^{\infty} \psi_n(y, z)\big(A_n\,\mathrm{e}^{-\mathrm{i}\kappa_n x} + B_n\,\mathrm{e}^{\mathrm{i}\kappa_n x}\big). \tag{195}$$

Each term in the series expansion, i.e. $\psi_n(y, z)\,\mathrm{e}^{-\mathrm{i}\kappa_n x}$, is called a duct mode. For hard walls, the eigenvalues α_n^2 are real and positive, except for the first one, which is $\alpha_1 = 0$. In the hard-wall case we have the important distinction between $k > \alpha_n$ where κ_n is real and the mode is propagating, and $k < \alpha_n$ where κ_n is imaginary and the mode is evanescent, i.e. exponentially decaying. Propagating modes are called "cut on", and evanescent modes are "cut off". The frequency $\omega = c_0\alpha_n$ is called the "cut-off frequency" of the mode. For low frequencies, the only cut-on mode is the plane wave (with cut-off frequency zero)

$$\psi_1 = 1, \qquad \alpha_1 = 0, \qquad \kappa_1 = \pm k.$$

The next eigenvalue α_2 is typically of the order of a number between 3 and 4 divided by the duct diameter D. As a result, any higher modes of a sound field with frequency $\omega < \sqrt{10}\,c_0/D$ decay faster than $\exp(-x/D)$. At any distance more than, say, two diameters away from a source, the field is well described by just plane waves.

If the duct cross section is circular or rectangular and the boundary condition is uniform everywhere, the solutions of the eigenvalue problem are relatively simple and may be found by separation of variables. These eigensolutions consist of combinations of trigonometric and Bessel functions in the circular case or combinations of trigonometric functions in the rectangular case. In particular for a cylindrical duct we have in polar coordinates (r, θ) the spiralling modes

$$\psi = J_m(\alpha_{m\mu} r)\,\mathrm{e}^{-\mathrm{i}m\theta}, \quad m \in \mathbb{Z}, \tag{196}$$

where J_m is a Bessel function of integer order m. Note that $J_{-m} = (-)^m J_m$. Positive m correspond with counter-clockwise rotating modes (phase $\omega t - m\theta =$ constant) and negative m with clockwise rotating modes. With a fixed source it is sometimes clearer to distinguish between symmetric and anti-symmetric modes by using $\sin(m\theta)$ and $\cos(m\theta)$ rather than $\mathrm{e}^{-\mathrm{i}m\theta}$. For a rectangular duct we have

$$\psi = \begin{Bmatrix} \sin(\beta_\mu y) \\ \cos(\beta_\mu y) \end{Bmatrix} \begin{Bmatrix} \sin(\gamma_\nu y) \\ \cos(\gamma_\nu y) \end{Bmatrix} \tag{197}$$

where $\alpha_{\mu\nu}^2 = \beta_\mu^2 + \gamma_\nu^2$. Note that due to the symmetry, in both geometries the eigenvalue has multiplicity 2: for each eigenvalue α there are 2 eigenfunctions. This is not the case for an arbitrary cross section. Some other geometries, like ellipses, do also allow explicit solutions, but only in special cases such as with hard walls. For other geometries one has to fall back on numerical methods for the eigenvalue problem.

Without mean flow the problem is symmetric, and to each eigenvalue there corresponds a right-running and a left-running mode, as both κ_n and $-\kappa_n$ can occur. The modes form a complete set of basis functions for the solutions to the wave equation. These modes are not exactly orthogonal to each other, but the complex conjugated modes (more precisely: the solutions of the adjoint – which is here the complex conjugated – problem) are mutually orthogonal or "bi-orthogonal" with ψ_n. It follows that if we denote $\hat{\psi}_n = \psi_n^*$ (the asterisk denotes the complex conjugate), we have

$$(\psi_n, \hat{\psi}_m) \stackrel{\text{def}}{=} \iint_{\mathcal{A}} \psi_n \hat{\psi}_m^* \, \mathrm{d}\sigma = \delta_{nm} \iint_{\mathcal{A}} \psi_n^2 \, \mathrm{d}\sigma = \delta_{nm} (\psi_n, \hat{\psi}_n), \tag{198}$$

$\delta_{nm} = 1$ if $n = m$ and is zero otherwise. Note that hard-walled modes are real, so $\hat{\psi}_n = \psi_n$ and thus hard-walled modes are orthogonal. The bi-orthogonality can be used to determine the coefficients of an expansion. Assume we have the pressure given along cross section $x = 0$ at the right side of a source, such that only right running modes occur. We have

$$p'(0, y, z) = F(y, z) = \sum_{n=1}^{\infty} A_n \psi_n(y, z). \tag{199}$$

After multiplication of left-hand and right-hand side by $\hat{\psi}_m$ and integration across $\mathcal{A}$ we get

$$A_m = \frac{(F, \hat{\psi}_m)}{(\psi_m, \hat{\psi}_m)}. \tag{200}$$

In a similar way we can determine an expansion of the Green's function $\hat{G}$ for a duct [81].

$$\hat{G}(x, y, z|\xi, \eta, \zeta) = \tfrac{1}{2}\mathrm{i} \sum_{n=1}^{\infty} \frac{\psi_n(y, z)\psi_n(\eta, \zeta)}{\kappa_n (\psi_n, \hat{\psi}_n)} \, \mathrm{e}^{-\mathrm{i}\kappa_n |x-\xi|} \, . \tag{201}$$

This result can also be obtained, albeit in a more laborious way, by means of Fourier transformation to x.

Solutions within more complex geometries, consisting of piecewise constant pipe elements, can be solved by matching two series of expansions in modes of adjacent pipe segments. This procedure involves an approximation by truncation of the mode expansion. This is quite successfully applied to stepwise changes in pipe cross-section [81, 60, 119], the diaphragm [81, 119], the elbow [74], the bend [12], the T-joint [107], the tone-hole of woodwinds [26]. A subtle point is the fact that the physical singularity of the sound field at the sharp edges is related to the convergence rate of the series expansion. It is therefore important, when using such approximations, to carefully tune the number of modes in the two series expansions [119].

With mean flow, the problem is not symmetric anymore. Only for uniform mean flow in a hard-walled duct this is only a minor problem, because the problem can be transformed to an equivalent no-flow problem, like we described in section 3.10.1. This is not possible in the combination of mean flow with soft walls, at least when we use the Ingard-Myers boundary condition [51, 87] for inviscid flow along an impedance wall, where the eigenvalue appears in the boundary condition. We have different eigenvalues for the left- and for the right-running modes, and we write instead of (195) now

$$p'(x, y, z) = \sum_{n=1}^{\infty} A_n \psi_n^+(y, z) \, \mathrm{e}^{-\mathrm{i}\kappa_n^+ x} + B_n \psi_n^-(y, z) \, \mathrm{e}^{-\mathrm{i}\kappa_n^- x} \, . \tag{202}$$

Orthogonality cannot be used to determine the amplitudes, but a Greens functions can still be determined by Fourier transformation to the axial coordinate. Furthermore, there are indications that the system may be unstable for certain impedances, with the result that one seemingly upstream-running mode is to be interpreted as really a downstream running instability[117].

Modes can also be obtained in more complex situations such as pipes with a main flow [29, 117, 98] or non-rigid wall's [113]. In many applications the prediction of mode propagation allows a significant insight into sound radiation problems. Example of such an application is the Tyler and Sofrin rules [134] for the design of aircraft turbines, minimizing the number of excited cut-on modes, and so the radiation of sound generated by rotor stator interaction.

When the duct and its possible mean flow vary in axial direction, modes that are strictly self-similar in x are in general not possible. However, if the duct varies slowly, like for example in an aero-engine flow duct, we can still identify approximate, "almost" modes, which retain their shape but with a slowly varying amplitude, eigenvalue and axial wave number [116, 118]. (The approximation is known as a variant of the WKB approximation.) If the duct and mean flow variations scale on a, so-called, slow variable $X = \varepsilon x$ where ε is small, we can describe a slowly varying mode by

$$p'(x, y, z) = N_n(X)\psi_n(y, z; X)\, \mathrm{e}^{-\mathrm{i}\int^x \kappa_n(\varepsilon\xi)\,\mathrm{d}\xi} \tag{203}$$

where ψ_n with κ_n is a mode with wave number at cross section $\mathcal{A}(X)$. The variable X acts as no more than a parameter. An adiabatic invariant can be identified yielding the varying amplitude. If the modes are normalised such that $(\psi_n, \hat{\psi}_n) = 1$ and the mean flow is nearly uniform, i.e. modulated by the duct, the amplitude N_n is given for hard-walled ducts by

$$\rho_0 N_n^2 \big(k^2 - (1 - M^2)\alpha_n^2\big)^{1/2} = \text{constant} \tag{204}$$

where ρ_0, $k = \omega/c_0$ and $M = U_0/c_0$ depend on X, and eigenvalue α_n corresponds to the mode ψ_n at position X. For soft walls the expression is similar. When the mode passes a turning point, i.e. where $k = (1 - M^2)^{1/2}\alpha_n$, the solution breaks down because the incident mode couples with its back-running counterpart (the mode "reflects"). A local analysis is possible to describe this effect [118, 95, 96, 97].

5.2 Low frequency Green's function in an infinitely long uniform duct

At frequencies below the cut-off frequency for higher order modes the acoustical field in a duct is, at some distance from the source, dominated by the plane wave mode. We expect therefore the Green's function to be independent of the transverse coordinate of the listener's position. Using the principle of reciprocity (65), this implies also that the Green's function G should not depend on the transverse coordinate of the source. Hence we can use a one dimensional Green's function g defined by:

$$\frac{1}{c_0^2}\frac{\partial^2 g}{\partial t^2} - \frac{\partial^2 g}{\partial x_3^2} = \delta(x_3 - y_3)\delta(t - \tau). \tag{205}$$

A solution of this equation will be nonzero only for $t > \tau$ and of the form $f(t - x_3/c_0)$ for $x_3 > y_3$ and $g(t + x_3/c_0)$ for $x_3 < y_3$. So we try (or see [59]) the auxiliary function

$$\varphi(x, t) = H(x)\, f\big(t - \tfrac{x}{c}\big) + H(-x) g\big(t + \tfrac{x}{c}\big),$$

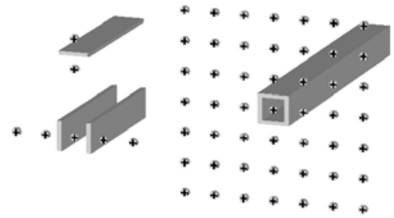

Figure 5: a) Method of images applied to a source at $\boldsymbol{y} = (y_1, y_2, y_3)$ at a distance y_3 from a hard wall $x_3 = 0$ has a Green's function: $G(\boldsymbol{x}, t|\boldsymbol{y}, \tau) = G_0(\boldsymbol{x}, t|\boldsymbol{y}, \tau) + G_0(\boldsymbol{x}, t|\boldsymbol{y}^*, \tau)$ with $\boldsymbol{y}^* = (y_1, y_2, -y_3)$. b) A source between two parallel hard walls generates an infinite row of images. c) A source in a rectangular duct generates an array of sources.

where H denotes the Heaviside stepfunction, and note that it has the properties

$$\frac{\partial^2\varphi}{\partial t^2} = H(x) f''\big(t - \tfrac{x}{c}\big) + H(-x) g''\big(t + \tfrac{x}{c}\big),$$

$$\frac{\partial\varphi}{\partial x} = \delta(x) f(t) - \delta(x) g(t) - \tfrac{1}{c} H(x) f'\big(t - \tfrac{x}{c}\big) + \tfrac{1}{c} H(-x) g'\big(t + \tfrac{x}{c}\big),$$

where we introduced $\delta(x) f(t - x/c) = \delta(x) f(t)$ and similarly for g,

$$\frac{\partial^2\varphi}{\partial x^2} = \delta'(x)(f(t) - g(t)) - \tfrac{1}{c}\delta(x)(f'(t) + g'(t)) + \tfrac{1}{c^2} H(x) f''\big(t - \tfrac{x}{c}\big) + \tfrac{1}{c^2} H(-x) g''\big(t + \tfrac{x}{c}\big),$$

and hence satisfies

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = -\delta'(x)(f(t) - g(t)) + \tfrac{1}{c}\delta(x)(f'(t) + g'(t)) = \delta(t)\delta(x).$$

if $f(t) = g(t) = \frac{1}{2} c H(t)$. By a simple coordinate transformation we find henceforth that

$$g(x_3, t|y_3, \tau) = \tfrac{1}{2} c_0 H\Big(t - \tau - \frac{|x_3 - y_3|}{c_0}\Big). \tag{206}$$

A more generic approach may be the use of Fourier transformation in t and x (see [119]).

This result can also be understood by using the method of images [102]. The acoustical field of a point source placed at a distance d from an infinite plane hard wall will generate the same acoustical field as the original source and it's image placed in free space (figure 5). The image's position is along the normal to the plane at the distance d from the plane, opposite to the source. The image takes into account the effect of the reflection of the waves on the plane. A source placed between two hard plane walls will generate an infinite row of images (figure 5). A source placed in a rectangular duct at $\boldsymbol{y} = (y_1, y_2, y_3)$ has the same effect as an array of sources in the (y_1, y_2) plane as shown in figure 5. Each source of the array generates a pulse $G_0 = \delta(t - t_e)/(4\pi r)$ with $t_e = t - r/c_0$. For a given listener coordinate $(0, 0, x_3)$ in the duct we have $r^2 = y_1^2 + y_2^2 + (x_3 - y_3)^2$ and $\mathrm{d}t_e = -\mathrm{d}r/c_0$. The number of images corresponding to an emission time between t_e and $t_e + \mathrm{d}t_e$ scales for fixed t, $\boldsymbol{x}$ and y_3 as $\pi\,\mathrm{d}(y_1^2 + y_2^2) = \pi\,\mathrm{d}r^2 = 2\pi r\,\mathrm{d}r = 2\pi c_0 r\,\mathrm{d}t_e$. As G_0 scales as r^{-1}, the increase in number of sources with r exactly compensates the decrease of G_0. This results into an Heaviside step function behaviour for a delta function source behaviour.

Note that for a source placed between two rigid planes we obtain a two dimensional response which is intermediate between the delta pulse and the Heaviside step. One observes a long decay scaling with $t^{-1/2}$. Thunder acts qualitatively as a line source with a two dimensional acoustical field. This phenomenon explains the long decay time of thunder sound. This makes acoustics in two dimensions quite complex and different from three dimensional and one dimensional acoustics [25]. A two dimensional acoustical field does not have a simple near-field behaviour as a three dimensional acoustical field. This essential difference between three dimensional and two dimensional acoustics is a major problem when considering the sound production by a two dimensional model of a three dimensional flow. Extending the two dimensional model to the acoustical far field will dramatically exaggerate radiation losses. Placing a radiation condition at a finite distance will provide results which depend on the distance between the boundary and the flow.

In an analogous way we may find that in the presence of a uniform subsonic main flow U_0 in the duct the Green's function, satisfying

$$\frac{1}{c_0^2}\Big(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\Big)^2 g-\frac{\partial^2 g}{\partial x^2}=\delta(x_3-y_3)\delta(t-\tau), \tag{207}$$

is given by

$$g(x_3,t|y_3,\tau)=\tfrac{1}{2}c_0 H(x_3-y_3)H\Big(t-\tau-\frac{x_3-y_3}{c_0+U_0}\Big)+\tfrac{1}{2}c_0 H(y_3-x_3)H\Big(t-\tau+\frac{x_3-y_3}{c_0-U_0}\Big). \tag{208}$$

Note that this Green's function satisfies the *reverse-flow* reciprocity principle [44] rather than (65). When we exchange the source and listener's positions, we should also reverse the main flow in order to keep the travel time $r/(c_0 \pm U_0)$ of the waves between the source and the listener constant.

5.3 Low frequency Green's function in a duct with a discontinuity

We want to introduce the concept of low frequency Green's function by considering the effect of a discontinuity in an infinitely long duct. We will use the reciprocity principle (65) in order to determine the Green's function. This implies that for a source placed at the discontinuity, we can deduce the Green's function by considering the sound field generated by a source placed at the position of the listener. For a source placed far from the discontinuity and a listener placed at the same side of the discontinuity as the source, the Green's function will be build up of the waves generated at the source plus the reflection of one of the waves at the discontinuity. A listener placed beyond the discontinuity will only be reached by the waves transmitted through the discontinuity. In other words the problem reduces to the determination of the reflected and transmitted waves at the discontinuity. We limit ourselves to the case of a compact transition region $x_I \le x_3 \le x_{II}$ of cross sectional area $S(x_3)$ between two semi-infinite ducts of cross section S_I at the side of the source and S_{II} at the opposite side. In the transition region we will now for simplicity assume a quasi-one dimensional incompressible potential flow $v_3'(x_3,t)$. Applying the integral (10a) mass conservation law across the discontinuity we have for $x_I \le x_3 \le x_{II}$:

$$S(x_3)v_3'(x_3,t)=S_I v_3'(x_I,t)=S_{II}v_3'(x_{II},t). \tag{209}$$

The equation of Bernoulli (42) applied in $x_I \le x_3 \le x_{II}$ between x_I and x_3 yields:

$$p'_I - p'(x_3,t) = \rho_0 \frac{\partial}{\partial t}\big[\phi(x_3,t) - \phi_I\big] = \rho_0 \frac{\partial}{\partial t}\Big[\int_{x_I}^{x_3} v'_3 \,\mathrm{d}x_3\Big]$$
$$= \rho_0 \Big[\int_{x_I}^{x_3} \frac{S_I}{S(x_3)}\,\mathrm{d}x_3\Big]\frac{\partial v'_3(x_I,t)}{\partial t}. \tag{210}$$

For harmonic waves $\hat{p}_I = p_I^+\,\mathrm{e}^{-\mathrm{i}kx_3} + p_I^-\,\mathrm{e}^{\mathrm{i}kx_3}$ and $\hat{p}_{II} = p_{II}^+\,\mathrm{e}^{-\mathrm{i}kx_3}$ we have:

$$S_I\big[p_I^+\,\mathrm{e}^{-\mathrm{i}kx_I} - p_I^-\,\mathrm{e}^{\mathrm{i}kx_I}\big] = S_{II}\,p_{II}^+\,\mathrm{e}^{-\mathrm{i}kx_{II}}$$

and:

$$p_I^+\,\mathrm{e}^{-\mathrm{i}kx_I} + p_I^-\,\mathrm{e}^{\mathrm{i}kx_I} - p_{II}^+\,\mathrm{e}^{-\mathrm{i}kx_{II}} = \mathrm{i}k_0 L_{\mathrm{eff}}\big[p_I^+\,\mathrm{e}^{-\mathrm{i}kx_I} - p_I^-\,\mathrm{e}^{\mathrm{i}kx_I}\big] \tag{211}$$

with the effective length L_{eff} defined by:

$$L_{\mathrm{eff}} = \int_{x_I}^{x_{II}} \frac{S_I}{S(x_3)}\,\mathrm{d}x_3. \tag{212}$$

We find for the reflection coefficient R and the transmission coefficient T:

$$R = \frac{p_I^-\,\mathrm{e}^{\mathrm{i}kx_I}}{P_I^+\,\mathrm{e}^{-\mathrm{i}kx_I}} = \frac{S_I - S_{II}(1 - \mathrm{i}kL_{\mathrm{eff}})}{S_I + S_{II}(1 + \mathrm{i}kL_{\mathrm{eff}})}$$

and:

$$T = \frac{p_{II}^+\,\mathrm{e}^{-\mathrm{i}kx_{II}}}{p_I^+\,\mathrm{e}^{-\mathrm{i}kx_I}}. \tag{213}$$

This results reduces to the well known result $R = [(S_I - S_{II})/(S_I + S_{II})]$ and $T = 2S_I/(S_I + S_{II})$ in the limit of $kL_{\mathrm{eff}} \to 0$. For an ideal open pipe end at $x_I = 0$ we find $R = -1$ by taking the limit $S_I/S_{II} \to 0$ and $x_I = 0$ [25, 82]. For a closed pipe at $x_I = 0$ we find $R = 1$ by taking the limit $S_I/S_{II} \to \infty$. When $S_I = S_{II}$ we recover the result for a diaphragm in a pipe $R = \mathrm{i}kL_{\mathrm{eff}}/(2+\mathrm{i}kL_{\mathrm{eff}})$ and $T = 2/(2 + \mathrm{i}kL_{\mathrm{eff}})$.

In the limit $kL_{\mathrm{eff}} \to 0$ the Green's function g which we are looking for is for $x_I > y_3$:

$$g = \frac{c_0}{2}H\Big(t - \tau + \frac{|x_3 - y_3|}{c_0}\Big) + R\frac{c_0}{2}H\Big(t - \tau + \frac{x_3 + y_3 - 2x_I}{c_0}\Big) \quad \text{for} \quad x_3 < x_I,$$

and:

$$g = T\frac{c_0}{2}H\Big(t - \tau - \frac{x_3 - y_3}{c_0}\Big) \quad \text{for} \quad x_I < x_3. \tag{214}$$

Considering the case $x_I > y_3$, while we neglect the effect of reflections $R = 0$ and $T = 1$, we can write for $kL_{\mathrm{eff}} \ll 1$ along $x_I < x_3 < x_{II}$

$$g = \frac{c_0}{2}H\Big(t - \tau - \frac{x_I + x_{\mathrm{eff}} - y_3}{c_0}\Big), \tag{215}$$

where:

$$x_{\mathrm{eff}} = \int_{x_I}^{x_3} \frac{S_I}{S(x_3)}\,\mathrm{d}x_3. \tag{216}$$

The effective coordinate x_{eff} corresponds to the potential difference induced between x_3 and x_I by a flow having at x_I a unit velocity $v'_I(x_I) = 1$.

We can use the same equations to the problem of an arbitrarily shaped discontinuity in a pipe, if we replace the definition (216) of the effective position, x_{eff}, by the more general definition:

$$x_{\mathrm{eff}} = \frac{\phi(x_3) - \phi_I}{v_I'}, \tag{217}$$

where we do not assume a quasi-one dimensional flow at the discontinuity. This corresponds to the low frequency Green's function proposed by Howe [44] for a discontinuity in a pipe. This generalisation allows to calculate the effect of a thin diaphragm in a pipe [82, 102] or the reflection at an elbow [10] by using potential theory for incompressible flows. This approach is limited to compact regions, but has the great advantage of providing a detailed model of the acoustical flow at such a discontinuity even in the neighborhood of sharp edges. Note that near a sharp edge an expansion of the solution in pipe modes fails to converge. Such theories are first order approximations in a Matched Asymptotic Expansion procedure [64]. The Match Asymptotic Expansion procedure allows a systematic approximation which can be extended to higher order in the small parameter kL_{eff}. In the present case we have neglected terms of order $(kL_{\mathrm{eff}})^2$. We take the inertia of the flow at the discontinuity into account but we neglect the effect of compressibility.

The same kind of approximation leads to the low frequency Green's function for a compact rigid body in free-space proposed by Howe [47, 48].

5.4 Aero-acoustics of an open pipe termination

5.4.1 Introduction

Modelling of the aero-acoustical behaviour of internal flow involves in many cases an open pipe termination. This is not only a boundary condition for the calculations of the internal acoustical field, the pipe termination is also a source of sound for the external acoustical field. In the presence of a main flow or at high amplitudes this involves a complex unsteady flow at the pipe termination. This problem is often oversimplified and underestimated. At low frequencies for example the boundary condition $p' = 0$ is often used. In the presence of a main flow this corresponds to a quasi-steady response of the free jet outside the pipe. This model is quite reasonable to describe the reflection of acoustical waves. When, however, we consider the convection of vorticity, such a model induces spurious sounds and flow phenomena due to the fact that vortices cannot be transported through such a boundary. Also such a model does not predict the spectacular effect of the shape of the nozzle edges on the aero-acoustical behaviour of the pipe termination. Under particular flow conditions and for specific nozzle shapes, the pipe termination can be a source of sound. Coupling with acoustical resonances of the pipe system results into whistling. We give here a short survey of the aero-acoustical behaviour of open pipe terminations.

5.4.2 Low frequency linear behaviour without main flow

We consider plane harmonic waves propagating in the x_1-direction along a duct of uniform cross sectional area $\mathcal{A}$:

$$p' = \hat{p}\,\mathrm{e}^{\mathrm{i}\omega t} = (p^+\,\mathrm{e}^{-\mathrm{i}kx_1} + p^-\,\mathrm{e}^{\mathrm{i}kx_1})\,\mathrm{e}^{\mathrm{i}\omega t}\,. \tag{218}$$

These waves induce a volume flow at the open pipe termination in x_1 with an amplitude:

$$\phi_V = \mathcal{A}\hat{v}_1 = \mathcal{A}\frac{p^+ - p^-}{\rho_0 c_0}. \tag{219}$$

The boundary condition at the open pipe termination $x_1 = 0$ for the internal acoustical field can be expressed in terms of a radiation impedance:

$$Z_p = \left(\frac{\hat{p}}{\hat{v}_1}\right)_{x_1=0} = \rho_0 c_0 \frac{p^+ + p^-}{p^+ - p^-}. \tag{220}$$

The acoustical power $< \mathcal{P} >$ radiated at the pipe termination is given by:

$$\langle \mathcal{P} \rangle = \langle (p' v_1')_{x_1=0} \rangle \mathcal{A} = \tfrac{1}{2}\hat{v}_1 \hat{v}_1^* \mathcal{A}\,\mathrm{Re}(Z_p) \tag{221}$$

where the asterisk denotes the complex conjugate. The real part of this impedance can be determined by applying the conservation of acoustical energy (equation 99) between the pipe exit and the far field outside the pipe. We consider first a thin walled, unflanged pipe, emerging into free space. In that case the far field will be dominated by the monopole radiation (82):

$$p' \simeq \frac{\mathrm{i}\omega\rho_0\phi_V}{4\pi r}\,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}kr} = \frac{\mathrm{i}\omega\rho_0\mathcal{A}\hat{v}_1}{4\pi r}\,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}kr}\,. \tag{222}$$

The corresponding radial velocity becomes for $kr \gg 1$ equal to $v_r' \simeq p'/(\rho_0 c_0)$. Together with eqaution (222) this gives for the power $\langle \mathcal{P} \rangle$ radiated in the far field

$$\langle \mathcal{P} \rangle = \langle p' v_r' \rangle 4\pi r^2 \simeq \frac{1}{\rho_0 c_0}\langle p'^2 \rangle 4\pi r^2 = \frac{1}{8\pi}\rho_0 c_0 k^2 \mathcal{A}^2 \hat{v}_1 \hat{v}_1^*. \tag{223}$$

Combination of equations (221) and (223)with the conservation of acoustical energy yields:

$$\mathrm{Re}(Z_p) = \rho_0 c_0 \frac{k^2\mathcal{A}}{4\pi}. \tag{224}$$

When we consider a pipe emerging from a hard wall into a half-infinite free space, flanged pipe, the radiation energy is distributed over a surface $2\pi r^2$ rather than $4\pi r^2$. Furthermore, the acoustical amplitude is increased by a factor two due to reflection of waves at the wall. The combination of both effects results into an increase of $\mathrm{Re}(Z_p)$ by a factor two compared to the unflanged pipe termination case (224). Further confinement of the pipe outlet will further increase the radiation impedance.

The imaginary part $\mathrm{Im}(Z_p)$ of the radiation impedance takes the inertia of the acoustical flow outside the pipe into account. This effect is often expressed in terms of an end correction δ:

$$\mathrm{Im}(Z_p) = \rho_0 c_0 k\delta. \tag{225}$$

This is the length of a pipe segment which has the same inertia as the outer flow. In first order approximation in the small parameter $k\sqrt{\mathcal{A}/\pi}$, the reflected wave p^- seems to be generated from a reflection of the incoming wave p^+, without phase shift, from an ideal open pipe termination with $p' = 0$ placed at $x_1 = \delta$. The pipe behaves acoustically if it were longer by a length δ if we assume that it is terminated by ideal open ends. In contrast with $\mathrm{Re}(Z_p)$ which is determined by global flow properties, the end correction δ of an open pipe termination is sensitive to the details of the local flow around at the pipe outlet. For a pipe with circular cross-section $\mathcal{A} = \pi a^2$ and for infinitely thin walls, Levine and Schwinger [66] found in the low frequency limit $\delta \simeq 0.61a$. For thick walls the end correction increases gradually from $\delta = 0.61a$ up to the flanged pipe limit $\delta = 0.82$ [1, 101, 91, 19]. Please note that the order of magnitude of the radiation impedance of an unflanged pipe is found by considering the low frequency limit of the radiation impedance Z_s of a compact sphere of radius a. Following equations (80) and (81) we have:

$$Z_s = \rho_0 c_0 \frac{\mathrm{i}ka + (ka)^2}{1 + (ka)^2} \simeq \rho_0 c_0 \big(\mathrm{i}ka + (ka)^2 + \dots\big). \tag{226}$$

The simple behaviour discussed above is typical for a radiation into a three dimensional free field. Confinement will drastically affect the radiation impedance. If we consider for example the radiation of a pipe termination emerging between two closely spaced parallel hard walls, the radiation field will be two dimensional. The radiation impedance has in such a case a complex behaviour [65]. One cannot for example identify a locally incompressible near field of the pipe termination. In general the radiation impedance will be much larger than in a three dimensional case. It is quite tempting to use a two dimensional flow simulation to describe the oscillating grazing flow along a wall cavity. It is however essential to realize that with this two dimensional calculation, we will dramatically overestimate the radiation losses of the flow. The results of the calculations will actually depend on the size of the calculation domain.

5.4.3 High frequency linear behaviour without main flow

For high frequencies any incident mode of amplitude A, frequency ω, circumferential order m and radial order μ reflects in several propagating radial modes. Due to cylindrical symmetry, no mode reflects in another circumferential order. Outside the pipe we have in the far field

$$p_{m\mu}(x, r) \simeq \rho_0 c_0^2 A D_{m\mu}(\xi) \frac{\mathrm{e}^{-\mathrm{i}k\varrho}}{k\varrho} \qquad (k\varrho \to \infty), \tag{227}$$

where $x = \varrho \cos\xi$, $r = \varrho \sin\xi$, and $D_{m\mu}(\xi)$ is called the directivity function, and $|D_{m\mu}(\xi)|$ is the radiation pattern. As each mode has its own spiralling phase plane orientation, the radiated pattern consists of lobes interlaced by zeros. Each zero is found at the propagation direction of a reflected mode, i.e. at $\xi = \arcsin(\alpha_{m\nu}/k)$, except for the direction of the incident mode, i.e. at $\xi_{m\mu} = \arcsin(\alpha_{m\mu}/k)$. As may be expected, here we find the radiation maximum or main lobe.

The field inside may be written as

$$p(x, r, \theta, t) = \rho_0 c_0^2 A \,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}m\theta} \Big[J_m(\alpha_{m\mu} r)\, \mathrm{e}^{-\mathrm{i}\kappa_{m\mu} x} + \sum_{\nu=1}^{\infty} R_{m\mu\nu} J_m(\alpha_{m\nu} r)\, \mathrm{e}^{\mathrm{i}\kappa_{m\nu} x} \Big] \tag{228}$$

where the branch of the square root

$$\kappa_{m\mu}^2 = \sqrt{k^2 - \alpha_{m\mu}^2}$$

is chosen such that $\kappa_{m\mu}$ is either positive real or negative imaginary. The matrix $\mathcal{R}_m = \{R_{m\mu\nu}\}$ is called the reflection matrix. Explicit analytical expressions (in the form of complex contour integrals) are found by application of the Wiener-Hopf method [66, 16].

A subtlety in the solution of the scattering problem is the duct edge. Here, the boundary condition is, technically speaking, not applicable, and without further condition we have no unique solution due to possible point or line sources "hiding" at the edge. So we have to add the (so-called) edge condition of finite energy in any neighbourhood of the edge. This amounts to an integrable squared velocity field $\sim |\nabla p|^2$. It transpires that p varies near the edge like the squareroot of the distance.

5.4.4 Influence of main flow on linear behaviour at low frequencies

Flow has a dramatic effect on the radiation impedance Z_p of an open pipe termination. We consider the effect of a uniform subsonic flow velocity U_0 in the pipe. In this section we limit our discussion to low frequencies. By low frequencies we do not only mean that we limit ourselves to plane-wave propagation in the pipe, we also assume that the flow at the pipe termination is locally quasi-steady. This corresponds to a low Strouhal number limit $St = \omega a/U_0 \ll 1$. As the Strouhal number is related to the Mach number $M = U_0/c_0$ by: $St = ka/M$, the combination of $St \ll 1$ with $M < 1$ implies that we consider very long wave length $ka \ll 1$.

The acoustic field in the pipe is described by the convective wave equation:

$$\Big(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x_1}\Big)^2 p' - c_0^2 \frac{\partial^2 p'}{\partial x_1^2} = 0. \qquad (229)$$

The solution of this wave equation is:

$$p' = p^+ \,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}k^+ x_1} + p^- \,\mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}k^- x_1} \qquad (230)$$

with:

$$k^\pm = \frac{\omega}{c_0 \pm U_0}. \qquad (231)$$

The fluctuations v_1' in the flow velocity are obtained by applying the linearized momentum equation $\rho_0(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x_1}) v_1' = -\frac{\partial}{\partial x_1} p'$:

$$v_1' = \frac{1}{\rho_0 c_0}\big(p^+ \,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}k^+ x_1} - p^- \,\mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}k^- x_1}\big) \qquad (232)$$

Following the Vortex Sound Theory [48], the acoustic field is most efficiently described in terms of fluctuation $B' = U_0 v_1' + p'/\rho_0$ of the total enthalpy $B = i + v_1^2/2$:

$$B' = B^+ \,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}k^+ x_1} + B^- \,\mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}k^- x_1} \qquad (233)$$

with:

$$B^\pm = p^\pm\Big(1 \pm \frac{U_0}{c_0}\Big). \qquad (234)$$

We consider a flow leaving the pipe. Due to flow separation at the pipe outlet, a free jet will be formed. It can be demonstrated that the only possible subsonic jet flow is a jet flow in which the pressure is uniform and equal to the pressure in the surroundings $p = p_{\mathrm{atm}}$ [128]. Following this model we have the boundary condition $p' = 0$ at the open pipe termination $x_1 = 0$. This implies a pressure reflection coefficient:

$$\frac{p^-}{p^+} = -1 \qquad (235)$$

and a total enthalpy reflection coefficient:

$$\frac{B^-}{B^+} = \frac{p^-}{p^+}\frac{c_0 - U_0}{c_0 + U_0} = -\frac{c_0 - U_0}{c_0 + U_0}. \tag{236}$$

This result implies a loss of acoustical energy. In the free jet the kinetic energy of the flow will be dissipated without any pressure recovery. As an acoustical modulation of the flow at the pipe termination implies a modulation of the kinetic energy in the jet, one can expect an absorption of acoustical energy. The vorticity in the shear layers of the free jet act as a source of sound. The end-correction appears to be determined by details of the flow just outside the pipe. As we will see in the next section in the limit of very low Strouhal numbers for an unflanged circular pipe $\delta \simeq 0.2a$ [109, 111, 101]. A rather unexpected result.

The above model corresponds to the very simple boundary condition $p' = 0$ at the pipe outlet $x_1 = 0$. While this model appears to be quite accurate in the limit $St \ll 1$, it should be used with care. A vortex with a dimension of the order of the pipe diameter correspond to a perturbation with a Strouhal number of order unity. A boundary condition of uniform pressure at a pipe termination will not allow such a vortex to flow out of the pipe! This is of course in contradiction with experimental observations, that vortices do leave the pipe through an open pipe termination.

In the case of inflow, we can have a potential flow. In such a case the radiation impedance for low frequencies will not be strongly affected by the flow. If flow separation occurs one can consider the use of a quasi-steady model [140].

It is interesting to note that a diaphragm placed at the pipe outlet can be used as a non-reflecting pipe termination at a critical Mach number $M = U_0/c_0$. This anechoic pipe termination behaviour was predicted by Bechert [4] using a quasi-steady flow model. Applying the quasi-steady equation of Bernoulli for an incompressible flow, between the pipe termination $x_1 = 0$ with cross section $\mathcal{A}$ and the free jet of cross section $\mathcal{A}_j$ formed at the diaphragm we have:

$$\tfrac{1}{2}\rho_0(U_0 + v_1')^2 + p(0,t) = \tfrac{1}{2}\rho_0(U_0 + v_1)^2\frac{\mathcal{A}}{\mathcal{A}_j}, \tag{237}$$

where we assume that in the jet the pressure remains constant, $p_j' = 0$. Substitution of $p' = p^+ + p^-$ and $v_1' = (p^+ - p^-)/(\rho_0 c_0)$ yields after linearisation:

$$\frac{p^+}{p^-} = -\frac{1 - \frac{U_0}{c_0}\left(\frac{\mathcal{A}^2}{\mathcal{A}_j^2} - 1\right)}{1 + \frac{U_0}{c_0}\left(\frac{\mathcal{A}^2}{\mathcal{A}_j^2} - 1\right)} \tag{238}$$

which vanishes for $U_0/c_0 = ((\mathcal{A}/\mathcal{A}_j)^2 - 1)^{-1}$. Generalisation of this result to arbitrary subsonic Mach numbers is discussed by Hofmans [42]. Application to the design of silencers is discussed by Durrieu [28].

When the Mach number is increased, we will eventually reach a critical flow at the diaphragm (choked flow, unit Mach number at the diaphragm). In this case the Mach number of a steady flow in the pipe is imposed by the ratio $\mathcal{A}/\mathcal{A}_j$ of pipe and diaphragm cross sections. It will in a quasi-steady theory remain constant, independent of any acoustical perturbation. The condition:

$$\frac{v_1'}{U_0} = \frac{c'}{c_0} \tag{239}$$

combined with the assumption of an ideal gas behaviour ($p'/p_0 = \gamma\rho'/\rho_0$ and $c_0^2 = \gamma p_0/\rho_0$) yields [2]:

$$\frac{p'}{v_1'} = \rho_0 c_0 \frac{2}{M(\gamma-1)}. \tag{240}$$

A more elaborated discussion of the acoustical response of a chocked nozzle is provided by Marbel and Candel [72].

5.4.5 Influence of main flow on linear behaviour at high frequencies

The available most complete model of sound radiation from a flow duct with jet, that is analytically tractable, is the following. It is a semi-infinite hard-walled duct with walls of zero thickness, containing inviscid uniform mean flow, density and sound speed, of which the values may differ inside and outside the duct or jet. The interface between jet and outer medium is modelled as a vortex sheet. Its analytically exact but mathematically rather involved solution of Wiener-Hopf type was given by Munt in [84, 85]. The solution contains all the elements of the no-flow solution, like reflection into radial modes and a radiation pattern with mode-related lobes, but the presence of flow adds a number of particular features. Of course, there are geometrical effects like a redirecting of the radiation pattern and refraction across the jet interface, but there are also two effects due to coupling with the mean flow.

First, for a velocity different across the vortex sheet, the jet is unstable. This instability is mathematically to be recognized by means of a causality analysis in the complex frequency plane. Second, there is the edge condition which is even more subtle than without flow (see section 3.10.3.) The condition of finite energy is still necessary to select physically possible solutions, but is now not enough for a unique solution. We still have a choice, corresponding to the amount of acoustic vorticity shed from the trailing edge. The possibility of vortex shedding is included in the model but its amplitude is not yet fixed, as it is determined by viscous and nonlinear processes which are not included. We have to add an extra condition. One such a condition is full regularity of the field at the trailing edge, the Kutta condition, which corresponds physically to the maximum amount of vortex shedding possible. Any other, "non-Kutta condition", solution will be singular at the edge but only one corresponds to no vortex shedding. Since it is the shed vorticity that excites the jet instability, the strict absence of the instability is a way to apply the condition of no vortex shedding for jets. If the mean flow is uniform everywhere, the absence of vortex shedding is easiest typified by a continuous potential. An alternative way to characterise the Kutta condition is via the streamline emanating from he edge, given by $r = a + \mathrm{Re}(h(x)\exp(\mathrm{i}\omega t - \mathrm{i}m\theta))$. Without Kutta condition its shape for $x \downarrow 0$ is like $h(x) = \mathcal{O}(x^{1/2})$. With Kutta condition it is like $h(x) = \mathcal{O}(x^{3/2})$.

The Kutta condition seems to be the relevant condition for relatively low frequencies and acoustic amplitudes and high Reynolds numbers [15]. Therefore it will be the condition that is supposed here throughout. Depending on the mean flow Reynolds number, dimensionless frequency and amplitude, other conditions (generalised Kutta conditions) are also possible [110, 112].

Physically, the shed vorticity, while moving near edge of the solid duct wall, produces some additional sound and thus adds some acoustic energy to the sound field. At the same time there is a certain amount of acoustic energy and a certain amount of mean flow energy needed for the creation of vorticity. Usually the net profit of energy is negative (acoustic energy is lost into the hydrodynamic vorticity) but this is not necessary [110].

Very interesting effects of vortex shedding occur for low frequencies, so we will consider results of Munt's solution in more detail in the next section.

5.4.6 Frequency dependence of the effect of flow on the radiation impedance

The experiments by Bechert, Michel and Pfizenmaier [5] in 1977 showed for the first time a dramatic loss of acoustic energy when a long plane wave from upstream a jet exhaust was partially reflected and partially transmitted at the exit. Although only a part was reflected, practically nothing was recovered in the radiated field outside. This was explained by Howe [45] in 1979 for low Mach numbers, by showing that due to the presence of mean flow the sound field sheds vorticity from the edge, in such a way that acoustic energy is converted into the acoustically undetected hydrodynamic energy. Cargill then showed in [13] that the Munt model [84] includes all the reported effects for any Mach number. This makes the Munt model extremely interesting for studying various aspects of sound field, mean flow coupling.

Let us consider the predictions of the Munt model for plane waves of relatively low wave number k in a duct of radius a with Mach jet number M_j. The feature that is directly related to the energy absorption is the modulus of the pressure reflection coefficient $|R|$. With Kutta condition, it tends to unity for low Helmholtz number ka, which yields for the transmitted power in the duct $\mathcal{P}_T \sim (1+M_j)^2 - |R|^2(1-M_j)^2$, a finite value, which contrasts the vanishing radiated value of $\mathcal{O}((ka)^2)$. The difference is the energy lost in the vortices.

Apart from the modulus reflection coefficient [85], little results are reported from the Munt solution pertaining to the low frequency range. Therefore, we present here some recent results obtained by P. in't Panhuis [71].

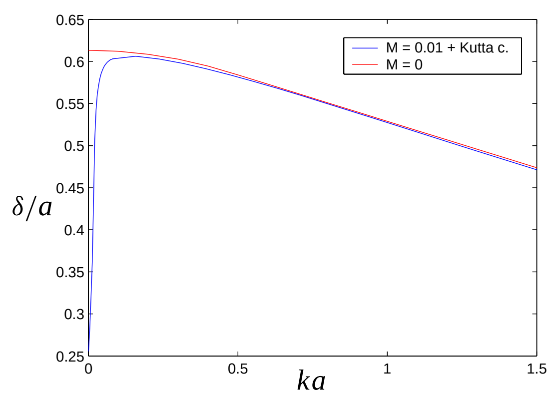

Figure 6: The endcorrection for no flow ($M_j = 0$) and a little flow ($M_j = 0.01$). Note the nonuniform behaviour for $ka \to 0$, $M_j \to 0$.

A rather remarkable feature is the behaviour of the end correction for a jet without co-flow. We define the end correction as the virtual point just outside the duct exit against which the plane wave appears to reflect with a free field boundary condition of: $|p|$ attains a minimum. If we write $R = -r\,\mathrm{e}^{-\mathrm{i}\theta}$, then the plane wave pressure is given by

$$p(x) \sim \mathrm{e}^{-\mathrm{i}\frac{kx}{1+M_j}} - r\,\mathrm{e}^{\mathrm{i}\frac{kx}{1+M_j} - \mathrm{i}\theta},$$

and so

$$|p(x)|^2 \sim 1 + r^2 - 2r\cos\left(\tfrac{2kM_j x}{1-M_j^2} - \theta\right)$$

which attains its minimum for $x = \delta$ where

$$\frac{\delta}{a} = \frac{(1-M_j^2)\theta}{2ka}.$$

A marked difference is the nonuniform limit of the end correction for $ka \to 0$ and $M_j \to 0$. As long as the Strouhal number ka/M_j is large, it converges to the no flow value ~ 0.6127, but once $ka \ll M_j$, the Strouhal number becomes small and the behaviour changes completely: the final value will be ~ 0.2554. Therefore, results for low Mach number and low Helmholtz number are usually better presented depending on the Strouhal number. See figure 6 where Munt's solution of very low Mach number is compared with the Levine and Schwinger [66] results for no flow. This non-uniform limit was predicted in [109, 111] and actually experimentally confirmed more than 10 years later by Peters et al. [101].

A pressure reflection coefficient larger than one for ka between 0 and, say, 1 was already reported by Munt [85]. In figure 7 the reflection coefficients together with the corresponding end corrections are given for various Mach numbers (no flow outside, no density or sound speed difference). Note that the limiting values for $ka \to 0$ of $|R|$ are unity for any Mach number; the endcorrections converge to $0.2554(1-M_j^2)^{1/2}$.

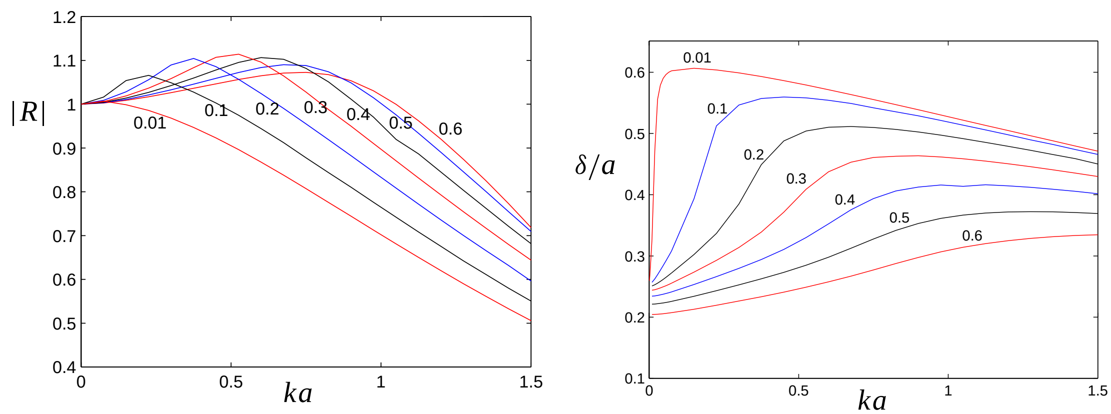

Figure 7: Plane wave reflection coefficient $|R|$ and endcorrection δ at jet exhaust without co-flow for $M_j = 0.01, 0.1, \ldots, 0.6$.

Finally, in figure 8 the effect of co-flow is shown. When the outer Mach number M_o varies from zero to the jet Mach number M_j, both reflection coefficient and end correction return to a behaviour similar to the no-flow case.

5.4.7 Whistling

Depending on the shape of the nozzle, a pipe termination can become a source of sound. This is a familiar phenomenon. By adjusting our lips and blowing at a critical velocity we can whistle.

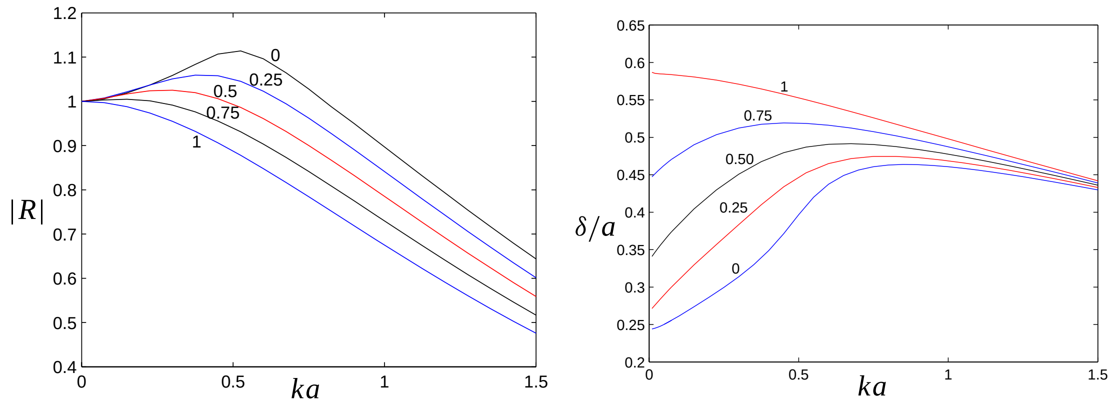

Figure 8: Plane wave reflection coefficient $|R|$ and end correction δ at jet exhaust with $M_j = 0.3$ and co-flow velocities $M_o/M_j = 0, 0.25, 0.5, 0.75, 1$.

Sound production by such a smoothly curved pipe outlet has been first studied in laboratory experiments by Powell [7] and by Wilson et. al [142]. We propose a qualitative model of this phenomenon in terms of Vortex Sound Theory. The same model has been used to explain the behaviour of whistler-nozzle [37, 39, 47]. The model has also been applied to other configurations in which self-sustained oscillations of the flow can be described in terms of discrete vortex shedding. Examples of such phenomena are the edge-tone [43], the Helmholtz resonator [92, 20, 77, 108], the splitter plate [139], the closed side branches [11, 143], the flute [135, 136, 21], the wall cavities [47, 133], the diffusors [67] and the solid propellant rocket engine [2]. Other examples of related self-sustained flow oscillations are described in many review papers and textbooks [120, 121, 6, 8].

Let us start our discussion by reconsidering the acoustical response of an unflanged pipe termination in the case of a uniform main flow U_0 directed towards the pipe outlet. The pipe cross section has a radius a. Flow separation at the sharp edges of the pipe outlet results into the formation of a free jet. We consider the reflection of harmonic plane waves with a frequency f travelling down the pipe, in the direction of the main flow. The acoustical flow $\mathbf{u}_{ac}$ is defined following the Vortex Sound Theory [46] as the unsteady part of the potential flow component of the flow field. This potential flow bends around the sharp edges of the pipe outlet as illustrated in figure 9.

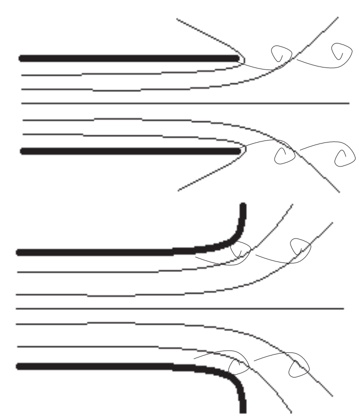

Figure 9: Acoustic flow at a pipe outlet a) for an unflanged pipe termination and b) for a horn.

In a potential flow the centripetal force on the fluid particles, allowing a curved stream line, is provided by the pressure gradient. The corresponding acceleration is $-|\mathbf{u}_{ac}|/R^2$ where R is the radius of curvature of the stream line. This implies that the pressure decreases towards the interior of the bend. As a consequence the velocity should increase. This is easily verified in a quasi-steady approximation by applying the equation of Bernoulli. At a sharp edge the radius of curvature of the stream lines in the potential flow vanish. This implies a singular behaviour. The pressure becomes infinitely low and the velocity infinitely large.

The singular behaviour of the flow at a sharp edge implies that viscous forces can never be neglected at such an edge. The result is that flow separation occurs which induces the shedding of vorticity. The vorticity is such that it compensates the potential flow singularity. Within the frame work of a frictionless theory this is the so-called Kutta condition [15]. At moderate acoustical amplitudes $|\mathbf{u}_{ac}|/U_0 \ll 1$, the amount of vorticity generated by the acoustic field will be negligible compared to the vorticity shedding induced by the main flow. The acoustic field does however trigger an instability of the shear layers of the free jet at the pipe exit. This instability results into the concentration of the vorticity into coherent structures which can be described as vortex rings. From experiments it appears that a new vortex start to be formed at the pipe outlet at the beginning of each acoustical oscillation period, when the acoustical velocity turns from pipe inwards to pipe outwards. We call this $t = 0$. The vortices travel along the pipe axis with a convective velocity U_c which is about half the main flow velocity U_0. The circulation of the vortex ring increases almost linearly in time as it accumulates the vorticity shed at the edge [92, 10, 21]. After one oscillation period $t = T$ a new vortex is shed an the vortex ring travels further downstream. Following the theory of Howe [46] the acoustical power generated by the vorticity is given by:

$$\boldsymbol{f}_c \cdot \mathbf{u}_{ac} = -\rho(\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{u}_{ac}. \tag{241}$$

This theory predicts that the vorticity field will initially absorb energy from the acoustic field. This seems logical as the acoustic field is perturbing the shear layer. This initial absorption is strong because, near the edge of the pipe exit, the acoustical velocity is normal to the convective velocity $\mathbf{v} = (U_c, 0, 0)$. Furthermore the acoustical velocity is large because of the singular behaviour near the sharp edge. The theory of Howe [46] predicts that after half an oscillation period $T/2 < t < T$ the vortex will produce sound because the acoustic flow is reversed while the signs of the convective velocity and the vorticity do not change. In spite of the growth of the vortex, the sound production will be weaker than the initial absorption which occurred for $0 < t < T/2$. This is due to the fact that the vortex moves away from the edge singularity. Both the magnitude of the acoustical velocity and the angle between the vortex path and the acoustical streamlines decrease. This discussion provides a qualitative understanding of the sound absorption predicted by linear theory in the previous sections.

An important feature of this discussion is that it stresses that the sound absorption is a balance between the initial absorption for $0 < t < T/2$ and the sound production for $T/2 < t < T$. This implies that it should be possible to obtain a net sound production by reducing the initial absorption. This is obtained by using a pipe with rounded edges like our lips or a horn. The singular behaviour at the flow separation point is suppressed. Furthermore the convective velocity near the flow separation point is almost parallel to the potential flow lines of the acoustic field. This reduces so much the initial absorption that at critical flow velocities sound production is observed. Typically one finds an optimal sound production when the travel time of the vortices along the lips corresponds to an oscillation period of the acoustic field. Experimentally one observes this for a Strouhal number based on the radius of curvature R of the lips given by: $St_R = fR/U_0 = 0.2$.

When acoustical energy can accumulate in a resonant mode of the pipe system upstream of the pipe termination one can observe self-sustained oscillation which are referred to as whistling. This occurs when the sound production is large enough to compensate for visco-thermal losses and radiation losses of the resonator. It is important to realize that in such a case the oscillation amplitude is limited by non-linear effects. This can be an increase of losses at high amplitudes or a saturation of the source. In the particular case of whistling the main non-linear amplitude limiting effect is the saturation of the circulation of the vortex rings. Once the vortex has accumulated all the vorticity available in the shear layer its circulation has reached a maximum value. In the case of a flute the main non-linear amplitude limiting mechanism is additional shedding of vorticity at sharp edges such as the labium [135]. The present example also illustrates that sound production does not necessarily involve impingement of a vortex or shear layer on a sharp edge.

This qualitative discussion indicates that for Strouhal number of order unity, a numerical flow simulation of a pipe system should include a model of the dynamical response of the free jet formed at a pipe outlet. Simple boundary conditions such as assuming a constant outlet pressure are only reasonable at low Strouhal numbers.

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